

Product formulas for the cyclotomic v -Schur algebra and for the canonical bases of the Fock space

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To Gus Lehrer on the occasion of his 60th birthday

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ABSTRACT. Let $\mathbf{F}_q[\mathbf{s}]$ be the q -deformed Fock space of level l with multi-charge $\mathbf{s} = (s_1, \dots, s_l)$. Uglov defined canonical bases $\mathcal{G}^\pm(\lambda, \mathbf{s})$ of $\mathbf{F}_q[\mathbf{s}]$ for l -partitions λ , and the polynomials $\Delta_{\lambda\mu}^\pm(q) \in \mathbb{Z}[q^\pm]$ are defined as the coefficients of the transition matrix between the canonical bases and the standard basis of $\mathbf{F}_q[\mathbf{s}]$. In this paper, we prove a product formula for $\Delta_{\lambda\mu}^\pm(q)$ for certain l -partitions λ, μ , which expresses $\Delta_{\lambda\mu}^\pm(q)$ as a product of such polynomials related to various smaller Fock spaces $\mathbf{F}_q[\mathbf{s}^{[i]}]$. Yvonne conjectures that $\Delta_{\lambda\mu}^+(q)$ are related to the q -decomposition numbers $d_{\lambda\mu}(q)$ of the cyclotomic v -Schur algebra $\mathcal{S}(A)$, where the parameters v, Q_1, \dots, Q_l are roots of unity in \mathbb{C} determined from \mathbf{s} . In our earlier work, we have proved a product formula for $d_{\lambda\mu}(q)$ of $\mathcal{S}(A)$, and the product formula for $\Delta_{\lambda\mu}^\pm(q)$ is regarded as a counter-part of that formula for the case of the Fock space.

0. INTRODUCTION

For a given integer $l > 0$, let Π^l be the set of l -partitions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$. Here $\lambda^{(i)} = (\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \dots)$ is a partition for $i = 1, \dots, l$. We denote by $|\lambda| = \sum_{i=1}^l |\lambda^{(i)}|$, and call it the size of λ , where $|\lambda^{(i)}| = \sum_j \lambda_j^{(i)}$. For given integers $l, n \geq 1$ and an indeterminate q , we consider the q -deformed Fock space $\mathbf{F}_q[\mathbf{s}]$ of level l with multi-charge $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}_{\geq 1}^l$, which is a vector space over $\mathbb{Q}(q)$ with the standard basis $\{|\lambda, \mathbf{s}\rangle \mid \lambda \in \Pi^l\}$, equipped with an action of the affine quantum group $U_q(\widehat{\mathfrak{sl}}_n)$. In [U], Uglov constructed the canonical bases $\mathcal{G}^\pm(\lambda, \mathbf{s})$ for $\mathbf{F}_q[\mathbf{s}]$ with respect to $U_q(\widehat{\mathfrak{sl}}_n)$, and for each $\lambda, \mu \in \Pi^l$, he defined a polynomial $\Delta_{\lambda\mu}^\pm(q) \in \mathbb{Q}[q^{\pm 1}]$ by the formula

$$\mathcal{G}^\pm(\lambda, \mathbf{s}) = \sum_{\mu \in \Pi^l} \Delta_{\lambda\mu}^\pm(q) |\mu, \mathbf{s}\rangle,$$

where $\Delta_{\lambda\mu}^\pm = 0$ unless $|\lambda| = |\mu|$.

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The cyclotomic v -Schur algebra $\mathcal{S}(\Lambda)$ with parameters v, Q_1, \dots, Q_l over a field R , associated to $\mathcal{H}_{N,l}$ was introduced by Dipper-James-Mathas [DJM], where $\mathcal{H}_{N,l}$ is the Ariki-Koike algebra associated to the complex reflection group $\mathfrak{S}_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$. Let Λ^+ be the set of $\lambda \in \Pi^l$ such that $|\lambda| = N$. $\mathcal{S}(\Lambda)$ is a cellular algebra in the sense of Graham-Lehrer [GL], and the Weyl module W^λ and its irreducible quotient L^λ are defined for each $\lambda \in \Lambda^+$. The main problem in the representation theory of $\mathcal{S}(\Lambda)$ is the determination of the decomposition numbers $[W^\lambda : L^\mu]$ for $\lambda, \mu \in \Lambda^+$. By making use of the Jantzen filtration of W^λ , the q -decomposition number $d_{\lambda\mu}(q)$ is defined, which is a polynomial analogue of the decomposition number, and we have $d_{\lambda\mu}(1) = [W^\lambda : L^\mu]$.

Let $\mathbf{p} = (l_1, \dots, l_g)$ be a g -tuple of positive integers such that $l_1 + \dots + l_g = l$. For each $\mu = (\mu^{(1)}, \dots, \mu^{(l)}) \in \Lambda^+$, one can associate a g -tuple of multi-partitions $(\mu^{[1]}, \dots, \mu^{[g]})$ by using \mathbf{p} , where $\mu^{[1]} = (\mu^{(1)}, \dots, \mu^{(l_1)})$, $\mu^{[2]} = (\mu^{(l_1+1)}, \dots, \mu^{(l_1+l_2)})$ and so on. We define $\alpha_{\mathbf{p}}(\mu) = (N_1, \dots, N_g)$, where $N_i = |\mu^{[i]}|$. Hence for $\lambda, \mu \in \Lambda^+$, $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$ means that $|\lambda^{[i]}| = |\mu^{[i]}|$ for $i = 1, \dots, g$. Let $\mathcal{S}(\Lambda_{N_i})$ be the cyclotomic v -Schur algebra associated to \mathcal{H}_{N_i, l_i} with parameters $v, Q_1^{[i]}, \dots, Q_{l_i}^{[i]}$, where $Q_1^{[1]} = Q_1, \dots, Q_{l_1}^{[1]} = Q_{l_1}, Q_1^{[2]} = Q_{l_1+1}, Q_2^{[2]} = Q_{l_1+2}, \dots$. In [SW], the product formula for the decomposition numbers of $\mathcal{S}(\Lambda)$ was proved, and it was extended in [W] to the product formula for q -decomposition numbers, which is given as follows; for $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have

$$(*) \quad d_{\lambda\mu}(q) = \prod_{i=1}^g d_{\lambda^{[i]}\mu^{[i]}}(q),$$

where $d_{\lambda^{[i]}\mu^{[i]}}(q)$ is the q -decomposition number for $\mathcal{S}(\Lambda_{N_i})$.

We assume that the parameters are given by $(v; Q_1, \dots, Q_l) = (\xi, \xi^{s_1}, \dots, \xi^{s_l})$, where ξ is a primitive n -th root of unity in \mathbb{C} and $\mathbf{s} = (s_1, \dots, s_l)$ is a multi-charge. For an integer M , we say that $|\lambda, \mathbf{s}|$ is M -dominant if $s_i - s_{i+1} > |\lambda| + M$ for $i = 1, \dots, l-1$. Yvonne [Y] gave a conjecture that $d_{\lambda\mu}(q)$ coincides with $\Delta_{\mu^\dagger \lambda^\dagger}^+(q)$ if $|\lambda, \mathbf{s}|$ is 0-dominant, where $\lambda^\dagger, \mu^\dagger$ are certain elements in Λ^+ induced from λ, μ (see Remark 2.7).

In view of Yvonne's conjecture, it is natural to expect a formula for $\Delta_{\lambda\mu}^+(q)$ as a counter-part for the Fock space of the product formula for $d_{\lambda\mu}(q)$. In fact, our result shows that it is certainly the case. We write $\mathbf{s} = (\mathbf{s}^{[1]}, \dots, \mathbf{s}^{[g]})$ with $\mathbf{s}^{[1]} = (s_1, \dots, s_{l_1})$, $\mathbf{s}^{[2]} = (s_{l_1+1}, \dots, s_{l_1+l_2})$, and so on. Let $\mathbf{F}_q[\mathbf{s}^{[i]}]$ be the q -deformed Fock space of level l_i with multi-charge $\mathbf{s}^{[i]}$ for $i = 1, \dots, g$. Then one can define polynomials $\Delta_{\lambda^{[i]}\mu^{[i]}}^\pm(q)$ for $\mathbf{F}_q[\mathbf{s}^{[i]}]$ similar to $\mathbf{F}_q[\mathbf{s}]$. The main result in this paper is the following product formula (Theorem 2.9); assume that $|\lambda, \mathbf{s}|$ is M -dominant for $M > 2n$. Then for $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have

$$(**) \quad \Delta_{\lambda\mu}^\pm(q) = \prod_{i=1}^g \Delta_{\lambda^{[i]}\mu^{[i]}}^\pm(q).$$

Yvonne's conjecture implies, by substituting $q = 1$, that $[W^\lambda : L^\mu] = \Delta_{\mu^\dagger \lambda^\dagger}^+(1)$, which we call LLT-type conjecture. In the case where $l = 1$, i.e., the case where $\mathcal{S}(\Lambda)$ is the v -Schur algebra of type A , it is known by Varagnolo-Vasserot [VV] that LLT-type conjecture holds. By applying the formulas (*), (**) to the case where $\mathbf{p} = (1, \dots, 1)$, we obtain the following partial result for the conjecture; we consider the general $\mathcal{S}(\Lambda)$, but assume that $|\lambda^{(i)}| = |\mu^{(i)}|$ for $i = 1, \dots, l$. Then LLT-type conjecture holds (under a stronger dominance condition) for $[W^\lambda : L^\mu]$.

This paper is organized as follows; in Section 1, we give a brief survey on the product formula for $\mathcal{S}(\Lambda)$ based on [SW], [W], which is a part of the first author's talk at the conference in Canberra, 2007. In Section 2 and 3, we prove the product formula (**) for the canonical bases of the Fock space.

1. PRODUCT FORMULA FOR THE CYCLOTOMIC v -SCHUR ALGEBRA

1.1. Let $\mathcal{H} = \mathcal{H}_{N,l}$ be the Ariki-Koike algebra over an integral domain R associated to the complex reflection group $W_{N,l} = \mathfrak{S}_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$ with parameters $v, Q_1, \dots, Q_l \in R$ such that v is invertible, which is an associative algebra with generators T_0, T_1, \dots, T_{N-1} and relations

$$\begin{aligned} (T_0 - Q_1) \cdots (T_0 - Q_l) &= 0 \\ (T_i - v)(T_i + v^{-1}) &= 0 \quad \text{for } i = 1, \dots, N-1 \end{aligned}$$

with other braid relations. The subalgebra generated by T_1, \dots, T_{N-1} is isomorphic to the Iwahori-Hecke algebra associated to the symmetric group \mathfrak{S}_N , and has a basis $\{T_w \mid w \in \mathfrak{S}_N\}$, where $T_w = T_{i_1} \cdots T_{i_r}$ for a reduced expression $w = s_{i_1} \cdots s_{i_r}$ of w .

An element $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{Z}_{\geq 0}^m$ is called a composition of $|\mu|$ consisting of m -parts, where $|\mu| = \sum_{i=1}^m \mu_i$. A composition μ is called a partition if $\mu_1 \geq \cdots \geq \mu_m \geq 0$. An l -tuple of compositions (resp. partitions) $\mu = (\mu^{(1)}, \dots, \mu^{(l)})$ is called an l -composition (resp. an l -partition) of $|\mu|$, where $|\mu| = \sum_i |\mu^{(i)}|$.

1.2. We define, following [DJM], a cyclotomic v -Schur algebra $\mathcal{S}(\Lambda)$ associated to \mathcal{H} . Fix $\mathbf{m} = (m_1, \dots, m_l) \in \mathbb{Z}_{>0}^l$ and let $\Lambda = \Lambda_{N,l}(\mathbf{m})$ (resp. $\Lambda^+ = \Lambda_{N,l}^+(\mathbf{m})$) be the set of l -compositions (resp. l -partitions) $\mu = (\mu^{(1)}, \dots, \mu^{(l)})$ of N such that $\mu^{(i)}$ has m_i -parts. For each $\mu \in \Lambda$, we define an element $m_\mu \in \mathcal{H}$ as follows; define $L_k \in \mathcal{H}$ ($1 \leq k \leq N$) by $L_1 = T_0$ and by $L_k = T_{k-1} L_{k-1} T_{k-1}$ for $k = 2, \dots, N$. Then L_1, \dots, L_N commute each other. For each $\mu \in \Lambda$, we define $\mathbf{a} = \mathbf{a}(\mu) = (a_1, \dots, a_l)$ by $a_k = \sum_{i=1}^{k-1} |\mu^{(i)}|$ for $k = 2, \dots, l$, and by $a_1 = 0$. Put

$$u_{\mathbf{a}}^+ = \prod_{k=1}^l \prod_{i=1}^{a_k} (L_i - Q_k), \quad x_\mu = \sum_{w \in \mathfrak{S}_\mu} v^{l(w)} T_w,$$

where $l(w)$ is the length of $w \in \mathfrak{S}_n$, and \mathfrak{S}_μ is the Young subgroup of \mathfrak{S}_n corresponding to μ . Then $u_{\mathbf{a}}^+$ commutes with x_μ , and we put $m_\mu = u_{\mathbf{a}}^+ x_\mu$.

For each $\mu \in \Lambda$, we define a right \mathcal{H} -module M^μ by $M^\mu = m_\mu \mathcal{H}$, and put $M = \bigoplus_{\mu \in \Lambda} M^\mu$. Then the cyclotomic q -Schur algebra $\mathcal{S}(\Lambda)$ is defined as

$$\mathcal{S}(\Lambda) = \text{End}_{\mathcal{H}} M = \bigoplus_{\nu, \mu \in \Lambda} \text{Hom}_{\mathcal{H}}(M^\mu, M^\nu).$$

1.3. For each $\lambda \in \Lambda^+$ and $\mu \in \Lambda$, the notion of semistandard tableau of shape λ and type μ was introduced by [DJM], extending the case of a partition λ and a composition μ . We denote by $\mathcal{T}_0(\lambda, \mu)$ the set of semistandard tableau of shape λ and type μ for $\lambda \in \Lambda^+, \mu \in \Lambda$. The notion of dominance order for partitions is also generalized for Λ , which we denote by $\mu \triangleleft \nu$. Note that $\mathcal{T}_0(\lambda, \mu)$ is empty unless $\lambda \trianglerighteq \mu$. We put $\mathcal{T}_0(\lambda) = \bigcup_{\mu \in \Lambda} \mathcal{T}_0(\lambda, \mu)$.

For each $S \in \mathcal{T}_0(\lambda, \mu), T \in \mathcal{T}_0(\lambda, \nu)$, Dipper-James-Mathas [DJM] constructed an \mathcal{H} -equivariant map $\varphi_{ST} : M^\nu \rightarrow M^\mu$. They showed that $\mathcal{S}(\Lambda)$ is a cellular algebra, in the sense of Graham-Lehrer [GL] with cellular basis

$$\mathcal{C}(\Lambda) = \{\varphi_{ST} \mid S, T \in \mathcal{T}_0(\lambda) \text{ for some } \lambda \in \Lambda^+\}.$$

1.4. Let $\mathbf{p} = (l_1, \dots, l_g)$ be a g -tuple of positive integers such that $\sum_{i=1}^g l_i = l$. For each $\mu = (\mu^{(1)}, \dots, \mu^{(l)})$, one can associate a g -tuple of multi-compositions $(\mu^{[1]}, \dots, \mu^{[g]})$ by making use of \mathbf{p} , where

$$\mu^{[1]} = (\mu^{(1)}, \dots, \mu^{(l_1)}), \quad \mu^{[2]} = (\mu^{(l_1+1)}, \dots, \mu^{(l_1+l_2)}), \quad \dots$$

For example assume that $N = 20, l = 5$ and $\mathbf{p} = (2, 2, 1)$. Take $\mu = (21; 121; 32; 1^3; 41)$. Then μ is written as $\mu = (\mu^{[1]}, \mu^{[2]}, \mu^{[3]})$ with

$$\mu^{[1]} = (21; 121), \quad \mu^{[2]} = (32; 1^3), \quad \mu^{[3]} = (41).$$

For $\mu = (\mu^{(1)}, \dots, \mu^{(l)}) = (\mu^{[1]}, \dots, \mu^{[g]}) \in \Lambda$, put

$$\alpha_{\mathbf{p}}(\mu) = (N_1, \dots, N_g), \quad \mathbf{a}_{\mathbf{p}}(\mu) = (a_1, \dots, a_g),$$

where $N_k = |\mu^{[k]}|$, and $a_k = \sum_{i=1}^{k-1} N_i$ for $k = 1, \dots, g$ with $a_1 = 0$.

Note that we often use the following relation. Take $\lambda = (\lambda^{[1]}, \dots, \lambda^{[g]})$, $\mu = (\mu^{[1]}, \dots, \mu^{[g]}) \in \Lambda$. Then $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$ if and only if $|\lambda^{[k]}| = |\mu^{[k]}|$ for $k = 1, \dots, g$.

1.5. Put

$$\begin{aligned} \mathcal{C}^{\mathbf{p}} &= \{\varphi_{ST} \in \mathcal{C}(\Lambda) \mid S \in \mathcal{T}_0(\lambda, \mu), T \in \mathcal{T}_0(\lambda, \nu), \\ &\quad \mathbf{a}_{\mathbf{p}}(\lambda) > \mathbf{a}_{\mathbf{p}}(\mu) \text{ if } \alpha_{\mathbf{p}}(\mu) \neq \alpha_{\mathbf{p}}(\nu), \mu, \nu \in \Lambda, \lambda \in \Lambda^+\}, \end{aligned}$$

where $\mathbf{a}_{\mathbf{p}}(\lambda) = (a_1, \dots, a_g) \geq \mathbf{a}_{\mathbf{p}}(\mu) = (b_1, \dots, b_g)$ if $a_k \geq b_k$ for $k = 1, \dots, g$, and $\mathbf{a}_{\mathbf{p}}(\lambda) > \mathbf{a}_{\mathbf{p}}(\mu)$ if $\mathbf{a}_{\mathbf{p}}(\lambda) \geq \mathbf{a}_{\mathbf{p}}(\mu)$ and $\mathbf{a}_{\mathbf{p}}(\lambda) \neq \mathbf{a}_{\mathbf{p}}(\mu)$. Let $\mathcal{S}^{\mathbf{p}}$ be the R -span of $\mathcal{C}^{\mathbf{p}}$. Then by using the cellular structure, one can show that $\mathcal{S}^{\mathbf{p}}$ is a subalgebra of $\mathcal{S}(\Lambda)$ containing the identity in $\mathcal{S}(\Lambda)$. The algebra $\mathcal{S}^{\mathbf{p}}$ turns out to be a standardly based

algebra in the sense of Du-Rui [DR] with respect to the poset $\Sigma^{\mathbf{p}}$, where $\Sigma^{\mathbf{p}}$ is a subset of $\Lambda^+ \times \{0, 1\}$ given by

$$\Sigma^{\mathbf{p}} = (\Lambda^+ \times \{0, 1\}) \setminus \{(\lambda, 1) \mid \mathcal{T}_0(\lambda, \mu) = \emptyset \text{ for any } \mu \in \Lambda \text{ such that } \mathbf{a}_{\mathbf{p}}(\lambda) > \mathbf{a}_{\mathbf{p}}(\mu)\},$$

and the partial order \geq on $\Sigma^{\mathbf{p}}$ is defined as $(\lambda_1, \varepsilon_1) > (\lambda_2, \varepsilon_2)$ if $\lambda_1 \triangleright \lambda_2$ or $\lambda_1 = \lambda_2$ and $\varepsilon_1 > \varepsilon_2$.

1.6. For each $\lambda \in \Lambda^+, \mu \in \Lambda$, we define a set $\mathcal{T}_0^{\mathbf{p}}(\lambda, \mu)$ by

$$\mathcal{T}_0^{\mathbf{p}}(\lambda, \mu) = \begin{cases} \mathcal{T}_0(\lambda, \mu) & \text{if } \mathbf{a}_{\mathbf{p}}(\lambda) = \mathbf{a}_{\mathbf{p}}(\mu), \\ \emptyset & \text{otherwise,} \end{cases}$$

and put $\mathcal{T}_0^{\mathbf{p}}(\lambda) = \bigcup_{\mu \in \Lambda} \mathcal{T}_0^{\mathbf{p}}(\lambda, \mu)$.

Let $\widehat{\mathcal{S}}^{\mathbf{p}}$ be the R -submodule of $\mathcal{S}^{\mathbf{p}}$ spanned by

$$\widehat{\mathcal{C}}^{\mathbf{p}} = \mathcal{C}^{\mathbf{p}} \setminus \{\varphi_{ST} \mid S, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda) \text{ for some } \lambda \in \Lambda^+\}.$$

Then $\widehat{\mathcal{S}}^{\mathbf{p}}$ turns out to be a two-sided ideal of $\mathcal{S}^{\mathbf{p}}$. We denote by $\overline{\mathcal{S}}^{\mathbf{p}} = \overline{\mathcal{S}}^{\mathbf{p}}(\Lambda)$ the quotient algebra $\mathcal{S}^{\mathbf{p}}/\widehat{\mathcal{S}}^{\mathbf{p}}$. Let $\pi : \mathcal{S}^{\mathbf{p}} \rightarrow \overline{\mathcal{S}}^{\mathbf{p}}$ be the natural projection, and put $\overline{\varphi} = \pi(\varphi)$ for $\varphi \in \mathcal{S}^{\mathbf{p}}$. One can show that $\overline{\mathcal{S}}^{\mathbf{p}}$ is a cellular algebra with cellular basis

$$\overline{\mathcal{C}}^{\mathbf{p}} = \{\overline{\varphi}_{ST} \mid S, T \in \mathcal{T}_0^{\mathbf{p}}(\lambda) \text{ for } \lambda \in \Lambda^+\}.$$

Thus we have constructed, for each \mathbf{p} , a subalgebra $\mathcal{S}^{\mathbf{p}}$ of $\mathcal{S}(\Lambda)$ and its quotient algebra $\overline{\mathcal{S}}^{\mathbf{p}}$. So we are in the following situation,

$$\begin{array}{ccc} \mathcal{S}^{\mathbf{p}} & \xrightarrow{\iota} & \mathcal{S}(\Lambda) \\ \pi \downarrow & & \\ \overline{\mathcal{S}}^{\mathbf{p}} & & \end{array}$$

where ι is an injection and π is a surjection.

Remark 1.7. In the special case where $\mathbf{p} = (1, \dots, 1)$, we have $g = l$, and $\mu^{[k]} = \mu^{(k)}$ for $k = 1, \dots, g = l$. Moreover in this case $\mathcal{S}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$ coincide with the subalgebra $\mathcal{S}^0(\mathbf{m}, N)$ and its quotient $\mathcal{S}(\mathbf{m}, N)$ considered in [SawS] in connection with the Schur-Weyl duality between \mathcal{H} and the quantum group $U_v(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{gl}_{m_1} \oplus \dots \oplus \mathfrak{gl}_{m_l}$ (cf. [SakS], [HS]). The other extreme case is $\mathbf{p} = (l)$, in which case $\mathcal{S}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$ coincide with $\mathcal{S}(\Lambda)$. Thus in general $\mathcal{S}^{\mathbf{p}}$ are regarded as intermediate objects between $\mathcal{S}(\Lambda)$ and $\mathcal{S}^0(\mathbf{m}, N)$.

1.8. In the rest of this section, unless otherwise stated we assume that R is a field. By a general theory of cellular algebras, one can define, for each $\lambda \in \Lambda^+$, the Weyl module W^λ and its irreducible quotient L^λ for $\mathcal{S}(\Lambda)$. Similarly, since $\overline{\mathcal{S}}^{\mathbf{p}}$ is a

cellular algebra, we have the Weyl module \overline{Z}^λ and its irreducible quotient \overline{L}^λ . Note that $\mathcal{S}(\Lambda)$ (resp. $\overline{\mathcal{S}}^{\mathbf{p}}$) is a quasi-hereditary algebra, and so the set $\{L^\lambda \mid \lambda \in \Lambda^+\}$ (resp. $\{\overline{L}^\lambda \mid \lambda \in \Lambda^+\}$) gives a complete set of representatives of irreducible $\mathcal{S}(\Lambda)$ -modules (resp. irreducible $\overline{\mathcal{S}}^{\mathbf{p}}$ -modules).

On the other hand, by using the general theory of standardly based algebras, one can construct, for each $\eta = (\lambda, \varepsilon) \in \Sigma^{\mathbf{p}}$, the Weyl module Z^η , and its irreducible quotient L^η (if it is non-zero). Thus the set $\{L^\eta \mid \eta \in \Sigma^{\mathbf{p}}, L^\eta \neq 0\}$ gives a complete set of representatives of irreducible $\mathcal{S}^{\mathbf{p}}$ -modules. In the case where $\eta = (\lambda, 0)$, we know more; $L^{(\lambda, 0)}$ is always non-zero for $\lambda \in \Lambda^+$, and the composition factors of $Z^{(\lambda, 0)}$ are all of the form $L^{(\mu, 0)}$ for some $\mu \in \Lambda^+$. We shall discuss the relations among the decomposition numbers

$$[W^\lambda : L^\mu]_{\mathcal{S}(\Lambda)}, \quad [Z^{(\lambda, 0)} : L^{(\mu, 0)}]_{\mathcal{S}^{\mathbf{p}}}, \quad [\overline{Z}^\lambda : \overline{L}^\mu]_{\overline{\mathcal{S}}^{\mathbf{p}}}.$$

(In order to distinguish the decomposition numbers for $\mathcal{S}(\Lambda)$, $\mathcal{S}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$, we use the subscripts such as $[W^\lambda : L^\mu]_{\mathcal{S}(\Lambda)}$). In the case where $\mathbf{p} = (1, \dots, 1)$, Sawada [Sa] discussed these relations. Our result below is a generalization of his result for the general \mathbf{p} .

1.9. First we consider the relation between the decomposition numbers of $\mathcal{S}^{\mathbf{p}}$ and that of $\overline{\mathcal{S}}^{\mathbf{p}}$. Under the surjection $\pi : \mathcal{S}^{\mathbf{p}} \rightarrow \overline{\mathcal{S}}^{\mathbf{p}}$, we regard an $\overline{\mathcal{S}}^{\mathbf{p}}$ -module as an $\mathcal{S}^{\mathbf{p}}$ -module. We have the following lemma.

Lemma 1.10. *For $\lambda, \mu \in \Lambda^+$, we have*

- (i) $\overline{Z}^\lambda \simeq Z^{(\lambda, 0)}$ as $\mathcal{S}^{\mathbf{p}}$ -modules.
- (ii) $\overline{L}^\mu \simeq L^{(\mu, 0)}$ as $\mathcal{S}^{\mathbf{p}}$ -modules.
- (iii) $[\overline{Z}^\lambda : \overline{L}^\mu]_{\overline{\mathcal{S}}^{\mathbf{p}}} = [Z^{(\lambda, 0)} : L^{(\mu, 0)}]_{\mathcal{S}^{\mathbf{p}}}$.
- (iv) Assume that $\alpha_{\mathbf{p}}(\lambda) \neq \alpha_{\mathbf{p}}(\mu)$. Then we have $[\overline{Z}^\lambda : \overline{L}^\mu]_{\overline{\mathcal{S}}^{\mathbf{p}}} = 0$.

1.11. Next we consider the relationship between the decomposition numbers of $\mathcal{S}^{\mathbf{p}}$ and that of $\mathcal{S}(\Lambda)$. Under the injection $\iota : \mathcal{S}^{\mathbf{p}} \hookrightarrow \mathcal{S}(\Lambda)$, we regard an $\mathcal{S}(\Lambda)$ -module as an $\mathcal{S}^{\mathbf{p}}$ -module. The following proposition was first proved in [Sa] in the case where $\mathbf{p} = (1, 1, \dots, 1)$. A similar argument works also for a general \mathbf{p} .

Proposition 1.12. *For each $\lambda \in \Lambda^+$, there exists an isomorphism of $\mathcal{S}(\Lambda)$ -modules*

$$Z^{(\lambda, 0)} \otimes_{\mathcal{S}^{\mathbf{p}}} \mathcal{S}(\Lambda) \simeq W^\lambda.$$

By using Lemma 1.10 and Proposition 1.12, we have the following theorem, which is a generalization of [Sa, Th. 5.7]. In fact in the theorem, the inequality

$$[Z^{(\lambda, 0)} : L^{(\mu, 0)}]_{\mathcal{S}^{\mathbf{p}}} \leq [W^\lambda : L^\mu]_{\mathcal{S}(\Lambda)}$$

always holds, and the converse inequality holds only when $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$.

Theorem 1.13 ([SW, Th. 3.13]). *For any $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have*

$$[\overline{Z}^\lambda : \overline{L}^\mu]_{\overline{\mathcal{S}}^{\mathbf{p}}} = [Z^{(\lambda,0)} : L^{(\mu,0)}]_{\mathcal{S}^{\mathbf{p}}} = [W^\lambda : L^\mu]_{\mathcal{S}(\Lambda)}.$$

1.14. In view of Theorem 1.13, the determination of the decomposition numbers $[W^\lambda : L^\mu]$ is reduced to that of the decomposition numbers $[\overline{Z}^\lambda : \overline{L}^\mu]_{\overline{\mathcal{S}}^{\mathbf{p}}}$ for $\overline{\mathcal{S}}^{\mathbf{p}}$ as far as the case where $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$. The algebra $\overline{\mathcal{S}}^{\mathbf{p}}$ has a remarkable structure as the following formula shows. In order to state our result, we prepare some notation. For each $N_k \in \mathbb{Z}_{\geq 0}$, put $\Lambda_{N_k} = \Lambda_{N_k, l_k}(\mathbf{m}^{[k]})$ and $\Lambda_{N_k}^+ = \Lambda_{N_k, l_k}^+(\mathbf{m}^{[k]})$. (Λ_{N_k} or $\Lambda_{N_k}^+$ is regarded as the empty set if $N_k = 0$.) For each $\mu^{[k]} \in \Lambda_{N_k}$, the \mathcal{H}_{N_k, l_k} -module $M^{\mu^{[k]}}$ is defined as in the case of the \mathcal{H} -module M^μ , and the cyclotomic q -Schur algebra $\mathcal{S}(\Lambda_{N_k})$ associated to the Ariki-Koike algebra \mathcal{H}_{N_k, l_k} is defined. The following theorem was first proved in [SawS] for $\mathbf{p} = (1, 1, \dots, 1)$ under a certain condition on parameters. Here we don't need any assumption on parameters.

Theorem 1.15 ([SW, Th. 4.15]). *There exists an isomorphism of R -algebras*

$$\overline{\mathcal{S}}^{\mathbf{p}} \simeq \bigoplus_{\substack{(N_1, \dots, N_g) \\ N_1 + \dots + N_g = N}} \mathcal{S}(\Lambda_{N_1}) \otimes \dots \otimes \mathcal{S}(\Lambda_{N_g}).$$

For $\lambda^{[k]}, \mu^{[k]} \in \Lambda_{N_k}^+$, let $W^{\lambda^{[k]}}$ be the Weyl module, and $L^{\mu^{[k]}}$ be the irreducible module with respect to $\mathcal{S}(\Lambda_{N_k})$. As a corollary to the theorem, we have

Corollary 1.16. *Let $\lambda, \mu \in \Lambda^+$. Then under the isomorphism in Theorem 1.15, we have the following.*

- (i) $\overline{Z}^\lambda \simeq W^{\lambda^{[1]}} \otimes \dots \otimes W^{\lambda^{[g]}}$.
- (ii) $\overline{L}^\mu \simeq L^{\mu^{[1]}} \otimes \dots \otimes L^{\mu^{[g]}}$.
- (iii) $[\overline{Z}^\lambda : \overline{L}^\mu]_{\overline{\mathcal{S}}^{\mathbf{p}}} = \begin{cases} \prod_{i=1}^g [W^{\lambda^{[i]}} : L^{\mu^{[i]}}]_{\mathcal{S}(\Lambda_{N_i})} & \text{if } \alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu), \\ 0 & \text{otherwise.} \end{cases}$

Combining this with Theorem 1.13, we have the following product formula for the decomposition numbers of $\mathcal{S}(\Lambda)$. The special case where $\mathbf{p} = (1, \dots, 1)$ is due to [Sa, Cor. 5.10], (still under a certain condition on parameters).

Theorem 1.17 ([SW, Theorem 4.17]). *For $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have*

$$[W^\lambda : L^\mu]_{\mathcal{S}(\Lambda)} = \prod_{i=1}^g [W^{\lambda^{[i]}} : L^{\mu^{[i]}}]_{\mathcal{S}(\Lambda_{N_i})}.$$

1.18 By making use of the Jantzen filtration, we shall define a polynomial analogue of the decomposition numbers, namely for each $\lambda, \mu \in \Lambda^+$, we define a polynomial $d_{\lambda\mu}(q) \in \mathbb{Z}[q]$ with indeterminate q such that $d_{\lambda\mu}(1)$ coincides with the decomposition number $[W^\lambda : L^\mu]_{\mathcal{S}(\Lambda)}$. We define similar polynomials also in the case for $\mathcal{S}^{\mathbf{p}}$ and $\overline{\mathcal{S}}^{\mathbf{p}}$.

We assume that R is a discrete valuation ring with the maximal ideal \mathfrak{p} , and let $F = R/\mathfrak{p}$ be the quotient field. We fix parameters $\widehat{v}, \widehat{Q}_1, \dots, \widehat{Q}_l$ in R , and let

$v, Q_1, \dots, Q_l \in F$ be their images under the natural map $R \rightarrow R/\mathfrak{p} = F$. Let $\mathcal{S}_R = \mathcal{S}_R(\Lambda)$ be the cyclotomic \widehat{v} -Schur algebra over R with parameters $\widehat{v}, \widehat{Q}_1, \dots, \widehat{Q}_l$, and $\mathcal{S} = \mathcal{S}(\Lambda)$ be the cyclotomic v -Schur algebra over F with parameters v, Q_1, \dots, Q_l . Thus $\mathcal{S} \simeq (\mathcal{S}_R + \mathfrak{p}\mathcal{S}_R)/\mathfrak{p}\mathcal{S}_R$. The algebras $\mathcal{S}_R^{\mathfrak{p}}, \overline{\mathcal{S}}_R^{\mathfrak{p}}$ over R , and the algebras $\mathcal{S}^{\mathfrak{p}}, \overline{\mathcal{S}}^{\mathfrak{p}}$ over F are defined as before. Let W_R^λ be the Weyl module of \mathcal{S}_R , and let $\langle \cdot, \cdot \rangle$ be the canonical bilinear form on W_R^λ arising from the cellular structure of $\mathcal{S}(\Lambda)_R$. For $i = 0, 1, \dots$, put

$$W_R^\lambda(i) = \{x \in W_R^\lambda \mid \langle x, y \rangle \in \mathfrak{p}^i \text{ for any } y \in W_R^\lambda\}$$

and define an F -vector space

$$W^\lambda(i) = (W_R^\lambda(i) + \mathfrak{p}W_R^\lambda)/\mathfrak{p}W_R^\lambda.$$

Then $W^\lambda(0) = W^\lambda$ is the Weyl module of \mathcal{S} , and we have a filtration

$$W^\lambda = W^\lambda(0) \supset W^\lambda(1) \supset W^\lambda(2) \supset \dots$$

of W^λ , which is the Jantzen filtration of W^λ .

Similarly, by using the cellular structure of $\overline{\mathcal{S}}^{\mathfrak{p}}$, and by using the property of the standardly based algebra of $\mathcal{S}^{\mathfrak{p}}$, one can define the Jantzen filtrations,

$$\begin{aligned} \overline{Z}^\lambda &= \overline{Z}^\lambda(0) \supset \overline{Z}^\lambda(1) \supset \overline{Z}^\lambda(2) \supset \dots, \\ Z^{(\lambda,0)} &= Z^{(\lambda,0)}(0) \supset Z^{(\lambda,0)}(1) \supset Z^{(\lambda,0)}(2) \supset \dots. \end{aligned}$$

Since W^λ (resp. $Z^{(\lambda,0)}, \overline{Z}^\lambda$) is a finite dimensional F -vector space, the Jantzen filtration gives a finite sequence. One sees that $W^\lambda(i)$ is an \mathcal{S} -submodule of W^λ by the associativity of the bilinear form, and similarly for $Z^{(\lambda,0)}$ and \overline{Z}^λ . Thus we define a polynomial $d_{\lambda\mu}(q)$ by

$$d_{\lambda\mu}(v) = \sum_{i \geq 0} [W^\lambda(i)/W^\lambda(i+1) : L^\mu] q^i,$$

where $[M : L^\mu] = [M : L^\mu]_{\mathcal{S}}$ denotes the multiplicity of L^μ in the composition series of the \mathcal{S} -module M as before. (In the notation below, we omit the subscripts \mathcal{S} , etc.) Similarly, we define, for $Z^{(\lambda,0)}$ and \overline{Z}^λ ,

$$\begin{aligned} d_{\lambda\mu}^{(\lambda,0)}(q) &= \sum_{i \geq 0} [Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i+1) : L^{(\mu,0)}] q^i, \\ \overline{d}_{\lambda\mu}(q) &= \sum_{i \geq 0} [\overline{Z}^\lambda(i)/\overline{Z}^\lambda(i+1) : \overline{L}^\mu] q^i. \end{aligned}$$

$d_{\lambda\mu}(q)$, $d_{\lambda\mu}^{(\lambda,0)}(q)$ and $\bar{d}_{\lambda\mu}(q)$ are polynomials in $\mathbb{Z}_{\geq 0}[q]$ and we call them q -decomposition numbers. Note that $d_{\lambda\mu}(1)$ coincides with $[W^\lambda : L^\mu]$, and similarly, we have $d_{\lambda\mu}^{(\lambda,0)}(1) = [Z^{(\lambda,0)} : L^{(\lambda,0)}]$, $\bar{d}_{\lambda\mu}(1) = [\bar{Z}^\lambda : \bar{L}^\mu]$.

As a q -analogue of Theorem 1.13 and Theorem 1.17, we have the following product formula for q -decomposition numbers.

Theorem 1.19 ([W, Th. 2.8, Th. 2.14]). *For $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$, we have*

$$d_{\lambda\mu}(q) = \bar{d}_{\lambda\mu}(q) = \prod_{i=1}^g d_{\lambda^{[i]}\mu^{[i]}}(q).$$

2. PRODUCT FORMULA FOR THE CANONICAL BASES OF THE FOCK SPACE

2.1. In the remainder of this paper, we basically follow the notation in Uglov [U]. First we review some notations. Fix positive integers n, l . Let $\Pi^l = \{\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})\}$ be the set of l -partitions. Take an l -tuple $\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$, and call it a multi-charge. Let $U_q(\widehat{\mathfrak{sl}}_n)$ be the quantum group of type $A_{n-1}^{(1)}$. The q -deformed Fock space $\mathbf{F}_q[\mathbf{s}]$ of level l with multi-charge \mathbf{s} is defined as a vector space over $\mathbb{Q}(q)$ with a basis $\{|\lambda, \mathbf{s}\rangle \mid \lambda \in \Pi^l\}$, equipped with a $U_q(\widehat{\mathfrak{sl}}_n)$ -module structure. The $U_q(\widehat{\mathfrak{sl}}_n)$ -module structure is defined as in [U, Th. 2.1], which depends on the choice of \mathbf{s} .

2.2. Put $s = s_1 + \dots + s_l$ for a multi-charge $\mathbf{s} = (s_1, \dots, s_l)$. Let $\mathbf{P}(s)$ be the set of semi-infinite sequences $\mathbf{k} = (k_1, k_2, \dots) \in \mathbb{Z}^\infty$ such that $k_i = s - i + 1$ for all $i \gg 1$, and $\mathbf{P}^{++}(s) = \{\mathbf{k} = (k_1, k_2, \dots) \in \mathbf{P}(s) \mid k_1 > k_2 > \dots\}$. For $\mathbf{k} \in \mathbf{P}(s)$, put $u_{\mathbf{k}} = u_{k_1} \wedge u_{k_2} \wedge \dots$, and call it a semi-infinite wedge. In the case where $\mathbf{k} \in \mathbf{P}^{++}(s)$ we call it an ordered semi-infinite wedge.

Let $\Lambda^{s+\frac{\infty}{2}}$ be a vector space over $\mathbb{Q}(q)$ spanned by $\{u_{\mathbf{k}} \mid \mathbf{k} \in \mathbf{P}(s)\}$ satisfying the ordering rule [U, Prop. 3.16]. By the ordering rule any semi-infinite wedge $u_{\mathbf{k}}$ can be written as a linear combination of some ordered semi-infinite wedges. It is known (cf. [U, Prop. 4.1]) that $\Lambda^{s+\frac{\infty}{2}}$ has a basis $\{u_{\mathbf{k}} \mid \mathbf{k} \in \mathbf{P}^{++}(s)\}$.

The vector space $\Lambda^{s+\frac{\infty}{2}}$ is called a semi-infinite wedge product. By [U, 4.2], $\Lambda^{s+\frac{\infty}{2}}$ has a structure of a $U_q(\widehat{\mathfrak{sl}}_n)$ -module. Let $\mathbb{Z}^l(s) = \{\mathbf{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l \mid s = \sum s_i\}$. Then we have

$$(2.2.1) \quad \Lambda^{s+\frac{\infty}{2}} \simeq \bigoplus_{\mathbf{s} \in \mathbb{Z}^l(s)} \mathbf{F}_q[\mathbf{s}] \quad \text{as } U_q(\widehat{\mathfrak{sl}}_n)\text{-modules.}$$

Thus we can regard $\mathbf{F}_q[\mathbf{s}]$ as a $U_q(\widehat{\mathfrak{sl}}_n)$ -submodule of $\Lambda^{s+\frac{\infty}{2}}$. The isomorphism in (2.2.1) is given through a bijection between two basis $\{u_{\mathbf{k}} \mid \mathbf{k} \in \mathbf{P}^{++}(s)\}$ and $\{|\lambda, \mathbf{s}\rangle \mid \lambda \in \Pi^l, \mathbf{s} \in \mathbb{Z}^l(s)\}$ as in [U, 4.1]. Identifying these bases, we write $u_{\mathbf{k}} = |\lambda, \mathbf{s}\rangle$ if $|\lambda, \mathbf{s}\rangle$ corresponds to $u_{\mathbf{k}}$.

2.3. For later use, we explain the explicit correspondence $u_{\mathbf{k}} \leftrightarrow |\lambda, \mathbf{s}\rangle$ given in [U, 4.1]. Assume given $u_{\mathbf{k}}$. Then for each $i \in \mathbb{Z}_{\geq 1}$, k_i is written as $k_i = a_i + n(b_i - 1) - nlm_i$, where $a_i \in \{1, \dots, n\}$, $b_i \in \{1, \dots, l\}$ and $m_i \in \mathbb{Z}$ are determined

uniquely. For $b \in \{1, \dots, l\}$, let $k_1^{(b)}$ be equal $a_i - nm_i$ where i is the smallest number such that $b_i = b$, and let $k_2^{(b)}$ be equal $a_j - nm_j$ where j is the next smallest number such that $b_j = b$, and so on. In this way, we obtain a strictly decreasing sequence $\mathbf{k}^{(b)} = (k_1^{(b)}, k_2^{(b)}, \dots)$ such that $k_i^{(b)} = s_b - i + 1$ for $i \gg 1$ for some uniquely determined integer s_b . Thus $\mathbf{k}^{(b)} \in \mathbf{P}^{++}(s_b)$, and one can define a partition $\lambda^{(b)} = (\lambda_1^{(b)}, \lambda_2^{(b)}, \dots)$ by $\lambda_i^{(b)} = k_i^{(b)} - s_b + i - 1$. We see that $\sum_b s_b = s$, and we obtain $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)})$ and $\mathbf{s} = (s_1, \dots, s_l)$. $u_{\mathbf{k}} \rightarrow |\lambda, \mathbf{s}\rangle$ gives the required bijection.

Note that the correspondence $u_{\mathbf{k}^{(b)}} \leftrightarrow |\lambda^{(b)}, s_b\rangle$ for each b is nothing but the correspondence $\Lambda^{s_b + \frac{\infty}{2}} \simeq \mathbf{F}_q[s_b]$ in the case where $\mathbf{F}_q[s_b]$ is a level 1 Fock space with charge s_b .

2.4. In [U], Uglov defined a bar-involution $\overline{}$ on $\Lambda^{s + \frac{\infty}{2}}$ by making use of the realization of the semi-infinite wedge product in terms of the affine Hecke algebra, which is semi-linear with respect to the involution $q \mapsto q^{-1}$ on $\mathbb{Q}(q)$, and commutes with the action of $U_q(\widehat{\mathfrak{sl}}_n)$, i.e., $\overline{u \cdot x} = \overline{u} \cdot \overline{x}$ for $u \in U_q(\widehat{\mathfrak{sl}}_n), x \in \Lambda^{s + \frac{\infty}{2}}$ (here \overline{u} is the usual bar-involution on $U_q(\widehat{\mathfrak{sl}}_n)$). We give a property of the bar-involution on $\Lambda^{s + \frac{\infty}{2}}$, which makes it possible to compute explicitly the bar-involution.

For $\mathbf{k} \in \mathbf{P}^{++}(s)$, we have

$$(2.4.1) \quad \overline{u_{\mathbf{k}}} = \overline{u_{k_1} \wedge u_{k_2} \wedge \dots \wedge u_{k_r}} \wedge u_{k_{r+1}} \wedge u_{k_{r+2}} \wedge \dots$$

for any $r \gg 1$. Moreover, for any (k_1, k_2, \dots, k_r) , not necessarily ordered, we have

$$(2.4.2) \quad \overline{u_{k_1} \wedge u_{k_2} \wedge \dots \wedge u_{k_r}} = \alpha(q) u_{k_r} \wedge \dots \wedge u_{k_2} \wedge u_{k_1}$$

with some $\alpha(q) \in \mathbb{Q}(q)$ of the form $\pm q^a$. The quantity $\alpha(q)$ is given explicitly as in [U, Prop. 3.23]. Thus one can express $\overline{u_{\mathbf{k}}}$ by the ordering rule as a linear combination of ordered semi-infinite wedges.

The bar-involution on $\Lambda^{s + \frac{\infty}{2}}$ leaves the subspace $\mathbf{F}_q[\mathbf{s}]$ invariant, and so defines a bar-involution on the Fock space $\mathbf{F}_q[\mathbf{s}]$. Let us define \mathcal{L}^+ (resp. \mathcal{L}^-) as the $\mathbb{Q}[q]$ -lattice (resp. $\mathbb{Q}[q^{-1}]$ -lattice) of $\Lambda^{s + \frac{\infty}{2}}$ generated by $\{|\lambda, \mathbf{s}\rangle \mid \lambda \in \Pi^l, \mathbf{s} \in \mathbb{Z}^l(s)\}$. Under this setting, Uglov constructed the canonical bases on $\mathbf{F}_q[\mathbf{s}]$.

Proposition 2.5 ([U, Prop. 4.11]). *There exist unique bases $\{\mathcal{G}^+(\lambda, \mathbf{s})\}, \{\mathcal{G}^-(\lambda, \mathbf{s})\}$ of $\mathbf{F}_q[\mathbf{s}]$ satisfying the following properties;*

- (i) $\overline{\mathcal{G}^+(\lambda, \mathbf{s})} = \mathcal{G}^+(\lambda, \mathbf{s}), \quad \overline{\mathcal{G}^-(\lambda, \mathbf{s})} = \mathcal{G}^-(\lambda, \mathbf{s}),$
- (ii) $\mathcal{G}^+(\lambda, \mathbf{s}) \equiv |\lambda, \mathbf{s}\rangle \pmod{q\mathcal{L}^+}, \quad \mathcal{G}^-(\lambda, \mathbf{s}) \equiv |\lambda, \mathbf{s}\rangle \pmod{q^{-1}\mathcal{L}^-},$

2.6. We define $\Delta_{\lambda, \mu}^{\pm}(q) \in \mathbb{Q}[q^{\pm 1}]$, for $\lambda, \mu \in \Pi^l$, by the formula

$$\mathcal{G}^{\pm}(\lambda, \mathbf{s}) = \sum_{\mu \in \Pi^l} \Delta_{\lambda, \mu}^{\pm}(q) |\mu, \mathbf{s}\rangle.$$

Note that $\Delta_{\lambda, \mu}^{\pm}(q) = 0$ unless $|\lambda| = |\mu|$.

For $\lambda \in \Pi^l$, $\mathbf{s} = (s_1, \dots, s_l)$, and $M \in \mathbb{Z}$, we say that $|\lambda, \mathbf{s}\rangle$ is M -dominant if $s_i - s_{i+1} > M + |\lambda|$ for $i = 1, \dots, l-1$.

Remark 2.7. Let $\mathcal{S}(A)$ be the cyclotomic v -Schur algebra over R with parameters v, Q_1, \dots, Q_l . We consider the special setting for parameters as follows; $R = \mathbb{C}$ and $(v; Q_1, \dots, Q_l) = (\xi; \xi^{s_1}, \dots, \xi^{s_l})$, where $\xi = \exp(2\pi i/n) \in \mathbb{C}$ and $\mathbf{s} = (s_1, \dots, s_l)$ is a multi-charge. For $\lambda = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \Pi^l$, we define an l -partition λ^\dagger by

$$\lambda^\dagger = ((\lambda^{(l)})', (\lambda^{(l-1)})', \dots, (\lambda^{(1)})'),$$

where $(\lambda^{(i)})'$ denotes the dual partition of the partition $\lambda^{(i)}$. Recall that $d_{\lambda\mu}(q) \in \mathbb{Z}[q]$ is the v -decomposition number defined in 1.18. In [Y], Yvonne gave the following conjecture;

Conjecture I: Assume that $|\lambda, \mathbf{s}|$ is 0-dominant. Then we have

$$d_{\lambda\mu}(q) = \Delta_{\mu^\dagger \lambda^\dagger}^+(q).$$

By specializing $q = 1$, Conjecture I implies an LLT-type conjecture for decomposition numbers of $\mathcal{S}(A)$,

Conjecture II: Under the same setting as in Conjecture I, we have

$$[W^\lambda : L^\mu]_{\mathcal{S}(A)} = \Delta_{\mu^\dagger \lambda^\dagger}^+(1).$$

In the case where $l = 1$, i.e., the case where $\mathcal{S}(A)$ is the v -Schur algebra associated to the Iwahori-Hecke algebra of type A , Conjecture II was proved by Varagnolo-Vasserot [VV]. It is open for the general case, $l > 1$. Concerning Conjecture I, it is not yet verified even in the case where $l = 1$.

2.8. Fix $\mathbf{p} = (l_1, \dots, l_g) \in \mathbb{Z}_{>0}^g$ such that $\sum_{i=1}^g l_i = l$ as in 1.4. For $i = 1, \dots, g$, define $\mathbf{s}^{[i]}$ by $\mathbf{s}^{[1]} = (s_1, \dots, s_{l_1})$, $\mathbf{s}^{[2]} = (s_{l_1+1}, \dots, s_{l_1+l_2})$, and so on. Thus we can write $\mathbf{s} = (\mathbf{s}^{[1]}, \dots, \mathbf{s}^{[g]})$. For each $\lambda \in \Pi^l$, we express it as $\lambda = (\lambda^{[1]}, \dots, \lambda^{[g]})$ as in 1.4. Recall the integer $\alpha_{\mathbf{p}}(\lambda)$ in 1.4. We have $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$ if and only if $|\lambda| = |\mu|$ and $|\lambda^{[i]}| = |\mu^{[i]}|$ for $i = 1, \dots, g$.

Let $\mathbf{F}_q[\mathbf{s}^{[i]}]$ be the q -deformed Fock space of level l_i with multi-charge $\mathbf{s}^{[i]}$, with basis $\{|\lambda^{[i]}, \mathbf{s}^{[i]}\rangle \mid \lambda^{[i]} \in \Pi^{l_i}\}$. We consider the canonical bases $\{\mathcal{G}^\pm(\lambda^{[i]}, \mathbf{s}^{[i]}) \mid \lambda^{[i]} \in \Pi^{l_i}\}$ of $\mathbf{F}_q[\mathbf{s}^{[i]}]$. Put

$$\mathcal{G}^\pm(\lambda^{[i]}, \mathbf{s}^{[i]}) = \sum_{\mu \in \Pi^{l_i}} \Delta_{\lambda^{[i]}, \mu^{[i]}}^\pm(q) |\mu^{[i]}, \mathbf{s}^{[i]}\rangle$$

with $\Delta_{\lambda^{[i]}, \mu^{[i]}}^\pm(q) \in \mathbb{Q}[q^{\pm 1}]$. The following product formula is our main theorem, which is a counter-part of Theorem 1.19 to the case of the Fock space, in view of Conjecture I.

Theorem 2.9. *Let $\lambda, \mu \in \Pi^l$ be such that $|\lambda, \mathbf{s}\rangle$ is M -dominant for $M > 2n$, and that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$. Then we have*

$$\Delta_{\lambda, \mu}^{\pm}(q) = \prod_{i=1}^g \Delta_{\lambda^{[i]}, \mu^{[i]}}^{\pm}(q).$$

As a corollary, we obtain a special case of Conjecture II (though we require a stronger dominance condition for $|\lambda, \mathbf{s}\rangle$).

Corollary 2.10. *Let $\lambda, \mu \in \Pi^l$ be such that $|\lambda^{(i)}| = |\mu^{(i)}|$ for $i = 1, \dots, l$. Assume that $|\lambda, \mathbf{s}\rangle$ is M -dominant for $M > 2n$. Then we have*

$$[W^{\lambda} : L^{\mu}]_{\mathcal{S}(A)} = \Delta_{\mu^{\dagger} \lambda^{\dagger}}^{+}(1).$$

Proof. Take $\mathbf{p} = (1, \dots, 1)$. Then we have $\lambda^{[i]} = \lambda^{(i)}$ for $i = 1, \dots, g = l$, and $\mathcal{S}(A_{N_i})$ coincides with the v -Schur algebra of type A . By applying Theorem 1.17 or Theorem 1.19, we have

$$[W^{\lambda} : L^{\mu}]_{\mathcal{S}(A)} = \prod_{i=1}^l [W^{\lambda^{(i)}} : L^{\mu^{(i)}}]_{\mathcal{S}(A_{N_i})}.$$

Also, by applying Theorem 2.9, we have

$$\Delta_{\mu^{\dagger} \lambda^{\dagger}}^{+}(1) = \prod_{i=1}^l \Delta_{(\mu^{(i)})'(\lambda^{(i)})'}^{+}(1).$$

On the other hand, we know $[W^{(\lambda^{(i)})} : L^{\mu^{(i)}}]_{\mathcal{S}(A_{N_i})} = \Delta_{(\mu^{(i)})'(\lambda^{(i)})'}^{+}(1)$ by a result of Valagnolo-Vasserot (see Remark 2.7). The corollary follows from these formulas. \square

2.11 Clearly the proof of the theorem is reduced to the case where $g = 2$, i.e., the case where $\lambda = (\lambda^{[1]}, \lambda^{[2]})$, etc. So, we assume that $\mathbf{p} = (l_1, l_2) = (t, l - t)$ for some $t \in \mathbb{Z}_{>0}$. We write the multi-charge \mathbf{s} as $\mathbf{s} = (\mathbf{s}^{[1]}, \mathbf{s}^{[2]})$, and consider the q -deformed Fock spaces $\mathbf{F}_q[\mathbf{s}^{[i]}]$ of level l_i with multi-charge $\mathbf{s}^{[i]}$ for $i = 1, 2$. We have an isomorphism $\mathbf{F}_q[\mathbf{s}] \simeq \mathbf{F}_q[\mathbf{s}^{[1]}] \otimes \mathbf{F}_q[\mathbf{s}^{[2]}]$ of vector spaces via the bijection of the bases $|\lambda, \mathbf{s}\rangle \leftrightarrow |\lambda^{[1]}, \mathbf{s}^{[1]}\rangle \otimes |\lambda^{[2]}, \mathbf{s}^{[2]}\rangle$ for each $\lambda = (\lambda^{[1]}, \lambda^{[2]}) \in \Pi^l$. For $\mathbf{s} \in \mathbb{Z}^l(s)$, put $s' = s_1 + \dots + s_t$, $s'' = s_{t+1} + \dots + s_l$. Under the isomorphism in (2.2.1) we have

$$\Lambda^{s' + \frac{\infty}{2}} \simeq \bigoplus_{\mathbf{s}^{[1]} \in \mathbb{Z}^{l_1}(s')} \mathbf{F}_q[\mathbf{s}^{[1]}], \quad \Lambda^{s'' + \frac{\infty}{2}} \simeq \bigoplus_{\mathbf{s}^{[2]} \in \mathbb{Z}^{l_2}(s'')} \mathbf{F}_q[\mathbf{s}^{[2]}].$$

Then we have an injective $\mathbb{Q}(q)$ -linear map

$$(2.11.1) \quad \Lambda^{s' + \frac{\infty}{2}} \otimes \Lambda^{s'' + \frac{\infty}{2}} \simeq \bigoplus_{\substack{\mathbf{s}^{[1]} \in \mathbb{Z}^{l_1}(s') \\ \mathbf{s}^{[2]} \in \mathbb{Z}^{l_2}(s'')}} \mathbf{F}_q[\mathbf{s}^{[1]}] \otimes \mathbf{F}_q[\mathbf{s}^{[2]}] \rightarrow \bigoplus_{\mathbf{s} \in \mathbb{Z}^l(s)} \mathbf{F}_q[\mathbf{s}] \simeq \Lambda^{s + \frac{\infty}{2}}$$

via $|\lambda^{[1]}, \mathbf{s}^{[1]}\rangle \otimes |\lambda^{[2]}, \mathbf{s}^{[2]}\rangle \mapsto |\lambda, \mathbf{s}\rangle$. We denote the embedding in (2.11.1) by Φ .

2.12. For $\lambda, \mu \in \Pi^l$, we define $\mathbf{a}(\lambda) > \mathbf{a}(\mu)$ if $|\lambda| = |\mu|$ and $|\lambda^{[1]}| > |\mu^{[1]}|$. Note that this is the same as the partial order $\mathbf{a}_{\mathbf{p}}(\lambda) > \mathbf{a}_{\mathbf{p}}(\mu)$ defined in 1.5 for the case where $\mathbf{p} = (l_1, l_2)$. We have the following proposition.

Proposition 2.13. *Assume that $u_{\mathbf{k}} = |\lambda, \mathbf{s}\rangle$ is M -dominant for $M > 2n$. Under the embedding $\Phi : \Lambda^{s'+\frac{\infty}{2}} \otimes \Lambda^{s''+\frac{\infty}{2}} \rightarrow \Lambda^{s+\frac{\infty}{2}}$ in 2.11, we have*

$$\overline{|\lambda, \mathbf{s}\rangle} = \overline{|\lambda^{[1]}, \mathbf{s}^{[1]}\rangle \otimes |\lambda^{[2]}, \mathbf{s}^{[2]}\rangle} + \sum_{\substack{\mu \in \Pi^l \\ \mathbf{a}(\lambda) > \mathbf{a}(\mu)}} \alpha_{\lambda, \mu} |\mu, \mathbf{s}\rangle$$

with $\alpha_{\lambda, \mu} \in \mathbb{Q}[q, q^{-1}]$.

2.14. Proposition 2.13 will be proved in 3.11 in the next section. Here assuming the proposition, we continue the proof of the theorem. We have the following result.

Theorem 2.15. *Assume that $|\lambda, \mathbf{s}\rangle = |\lambda^{[1]}, \mathbf{s}^{[1]}\rangle \otimes |\lambda^{[2]}, \mathbf{s}^{[2]}\rangle$ is M -dominant for $M > 2n$. Then we have*

$$\begin{aligned} \mathcal{G}^{\pm}(\lambda, \mathbf{s}) &= \mathcal{G}^{\pm}(\lambda^{[1]}, \mathbf{s}^{[1]}) \otimes \mathcal{G}^{\pm}(\lambda^{[2]}, \mathbf{s}^{[2]}) \\ &\quad + \sum_{\substack{\mu \in \Pi^l \\ \mathbf{a}(\lambda) > \mathbf{a}(\mu)}} b_{\lambda, \mu}^{\pm} \mathcal{G}^{\pm}(\mu^{[1]}, \mathbf{s}^{[1]}) \otimes \mathcal{G}^{\pm}(\mu^{[2]}, \mathbf{s}^{[2]}) \end{aligned}$$

with $b_{\lambda, \mu}^{\pm} \in \mathbb{Q}[q^{\pm 1}]$.

Proof. Throughout the proof, we write $\Delta_{\lambda^{[i]}, \mu^{[i]}}^{\pm}(q)$ as $\Delta_{\lambda^{[i]}, \mu^{[i]}}^{\pm}$ for simplicity. Since

$$\begin{aligned} &\mathcal{G}^{\pm}(\lambda^{[1]}, \mathbf{s}^{[1]}) \otimes \mathcal{G}^{\pm}(\lambda^{[2]}, \mathbf{s}^{[2]}) \\ &= \sum_{\mu^{[1]} \in \Pi^{l_1}} \Delta_{\lambda^{[1]}, \mu^{[1]}}^{\pm} |\mu^{[1]}, \mathbf{s}^{[1]}\rangle \otimes \sum_{\mu^{[2]} \in \Pi^{l_2}} \Delta_{\lambda^{[2]}, \mu^{[2]}}^{\pm} |\mu^{[2]}, \mathbf{s}^{[2]}\rangle \\ &= \sum_{\mu \in \Pi^l} \Delta_{\lambda^{[1]}, \mu^{[1]}}^{\pm} \Delta_{\lambda^{[2]}, \mu^{[2]}}^{\pm} |\mu, \mathbf{s}\rangle, \end{aligned}$$

we have, by Proposition 2.13,

$$\begin{aligned} &\overline{\mathcal{G}^{\pm}(\lambda^{[1]}, \mathbf{s}^{[1]}) \otimes \mathcal{G}^{\pm}(\lambda^{[2]}, \mathbf{s}^{[2]})} \\ &= \sum_{\mu \in \Pi^l} \overline{\Delta_{\lambda^{[1]}, \mu^{[1]}}^{\pm} \Delta_{\lambda^{[2]}, \mu^{[2]}}^{\pm} |\mu, \mathbf{s}\rangle} \\ &= \sum_{\mu \in \Pi^l} \overline{\Delta_{\lambda^{[1]}, \mu^{[1]}}^{\pm} \Delta_{\lambda^{[2]}, \mu^{[2]}}^{\pm} \left\{ \overline{|\mu^{[1]}, \mathbf{s}^{[1]}\rangle \otimes |\mu^{[2]}, \mathbf{s}^{[2]}\rangle} + \sum_{\substack{\nu \in \Pi^l \\ \mathbf{a}(\mu) > \mathbf{a}(\nu)}} \alpha_{\mu, \nu} |\nu, \mathbf{s}\rangle \right\}} \\ &= \overline{\mathcal{G}^{\pm}(\lambda^{[1]}, \mathbf{s}^{[1]})} \otimes \overline{\mathcal{G}^{\pm}(\lambda^{[2]}, \mathbf{s}^{[2]})} \end{aligned}$$

$$+ \sum_{\mu \in \Pi^l} \overline{\Delta_{\lambda^{[1]}, \mu^{[1]}}^{\pm}} \overline{\Delta_{\lambda^{[2]}, \mu^{[2]}}^{\pm}} \left\{ \sum_{\substack{\nu \in \Pi^l \\ \mathbf{a}(\mu) > \mathbf{a}(\nu)}} \alpha_{\mu, \nu} |\nu^{[1]}, \mathbf{s}^{[1]}\rangle \otimes |\nu^{[2]}, \mathbf{s}^{[2]}\rangle \right\}.$$

By the property of the canonical bases, we have $\overline{\mathcal{G}^{\pm}(\lambda^{[i]}, \mathbf{s}^{[i]})} = \mathcal{G}^{\pm}(\lambda^{[i]}, \mathbf{s}^{[i]})$ for $i = 1, 2$. Note that, $|\lambda^{[i]}| = |\mu^{[i]}|$ if $\Delta_{\lambda^{[i]}, \mu^{[i]}}^{\pm} \neq 0$ for $i = 1, 2$. Moreover, a vector $|\nu^{[i]}, \mathbf{s}^{[i]}\rangle$ can be written as a linear combination of the canonical bases $\mathcal{G}^{\pm}(\kappa^{[i]}, \mathbf{s}^{[i]})$ such that $|\kappa^{[i]}| = |\nu^{[i]}|$ for $i = 1, 2$. Hence we have

$$(2.15.1) \quad \begin{aligned} & \overline{\mathcal{G}^{\pm}(\lambda^{[1]}, \mathbf{s}^{[1]})} \otimes \overline{\mathcal{G}^{\pm}(\lambda^{[2]}, \mathbf{s}^{[2]})} \\ &= \mathcal{G}^{\pm}(\lambda^{[1]}, \mathbf{s}^{[1]}) \otimes \mathcal{G}^{\pm}(\lambda^{[2]}, \mathbf{s}^{[2]}) + \sum_{\substack{\mu \in \Pi^l \\ \mathbf{a}(\lambda) > \mathbf{a}(\mu)}} b'_{\lambda, \mu}^{\pm} \mathcal{G}^{\pm}(\mu^{[1]}, \mathbf{s}^{[1]}) \otimes \mathcal{G}^{\pm}(\mu^{[2]}, \mathbf{s}^{[2]}) \end{aligned}$$

with $b'_{\lambda, \mu}^{\pm} \in \mathbb{Q}[q, q^{-1}]$. Thus one can write as

$$\overline{\mathcal{G}^{\pm}(\lambda^{[1]}, \mathbf{s}^{[1]})} \otimes \overline{\mathcal{G}^{\pm}(\lambda^{[2]}, \mathbf{s}^{[2]})} = \sum_{\substack{\mu \in \Pi^l \\ |\mu| = |\lambda|}} R_{\lambda, \mu}^{\pm} \mathcal{G}^{\pm}(\mu^{[1]}, \mathbf{s}^{[1]}) \otimes \mathcal{G}^{\pm}(\mu^{[2]}, \mathbf{s}^{[2]}),$$

with $R_{\lambda, \mu}^{\pm} \in \mathbb{Q}[q, q^{-1}]$, where the matrix $(R_{\lambda, \mu}^{\pm})_{|\lambda|=|\mu|}$ is unitriangular with respect to the order compatible with $\mathbf{a}(\lambda) > \mathbf{a}(\mu)$ by (2.15.1). Thus, by a standard argument for constructing the canonical bases, we have

$$\begin{aligned} \mathcal{G}^{\pm}(\lambda, \mathbf{s}) &= \mathcal{G}^{\pm}(\lambda^{[1]}, \mathbf{s}^{[1]}) \otimes \mathcal{G}^{\pm}(\lambda^{[2]}, \mathbf{s}^{[2]}) \\ &+ \sum_{\substack{\mu \in \Pi^l \\ \mathbf{a}(\lambda) > \mathbf{a}(\mu)}} b_{\lambda, \mu}^{\pm} \mathcal{G}^{\pm}(\mu^{[1]}, \mathbf{s}^{[1]}) \otimes \mathcal{G}^{\pm}(\mu^{[2]}, \mathbf{s}^{[2]}), \end{aligned}$$

with $b_{\lambda, \mu}^{\pm} \in \mathbb{Q}[q^{\pm 1}]$. This proves Theorem 2.15. \square

2.16. We now prove Theorem 2.9 in the case where $g = 2$, assuming that Proposition 2.13 holds. Assume that $|\lambda, \mathbf{s}\rangle$ is M -dominant for $M > 2n$. Since $\mathcal{G}(\lambda^{[i]}, \mathbf{s}^{[i]}) = \sum_{\mu^{[i]} \in \Pi^{l_i}} \Delta_{\lambda^{[i]}, \mu^{[i]}}^{\pm} |\mu^{[i]}, \mathbf{s}^{[i]}\rangle$, it follows from Theorem 2.15 that

$$\mathcal{G}^{\pm}(\lambda, \mathbf{s}) = \sum_{\substack{\mu \in \Pi^l \\ \alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)}} \Delta_{\lambda^{[1]}, \mu^{[1]}}^{\pm} \Delta_{\lambda^{[2]}, \mu^{[2]}}^{\pm} |\mu, \mathbf{s}\rangle + \sum_{\substack{\mu \in \Pi^l \\ \mathbf{a}(\lambda) > \mathbf{a}(\mu)}} \tilde{b}_{\lambda, \mu}^{\pm} |\mu, \mathbf{s}\rangle.$$

Since $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$ is equivalent to $\mathbf{a}(\lambda) = \mathbf{a}(\mu)$, this implies that for any $\mu \in \Pi^l$ such that $\alpha_{\mathbf{p}}(\lambda) = \alpha_{\mathbf{p}}(\mu)$,

$$(2.16.1) \quad \Delta_{\lambda, \mu}^{\pm}(q) = \Delta_{\lambda^{[1]}, \mu^{[1]}}^{\pm}(q) \Delta_{\lambda^{[2]}, \mu^{[2]}}^{\pm}(q).$$

This proves Theorem 2.9 modulo Proposition 2.13.

Remark 2.17. By [U, Th. 3.26], $\Delta_{\lambda\mu}^{\pm}(q)$ can be interpreted by parabolic Kazhdan-Lusztig polynomials of an affine Weyl group. So, Theorem 2.9 gives a product formula for parabolic Kazhdan-Lusztig polynomials. It would be interesting to give a geometric interpretation of this formula.

3. TENSOR PRODUCT OF THE FOCK SPACES

3.1. In this section, we prove Proposition 2.13 after some preliminaries. The proof will be given in 3.11. For a given $u_{\mathbf{k}} = |\lambda, \mathbf{s}\rangle$, we associate semi-infinite sequences $\mathbf{k}^{(b)} = (k_1^{(b)}, k_2^{(b)}, \dots) \in \mathbf{P}^{++}(s_b)$ for $b = 1, \dots, l$ as in 2.3. For $b \in \{1, \dots, l\}$ and $k \in \mathbb{Z}$, we put

$$(3.1.1) \quad u_k^{(b)} = u_{a+n(b-1)-nlm},$$

where $a \in \{1, \dots, n\}$ and $m \in \mathbb{Z}$ are uniquely determined by $k = a - nm$. For a positive integer r , put $\mathbf{k}_r = (k_1, k_2, \dots, k_r)$. Then $u_{\mathbf{k}}$ can be written as $u_{\mathbf{k}} = u_{\mathbf{k}_r} \wedge u_{k_{r+1}} \wedge u_{k_{r+2}} \wedge \dots$, where $u_{\mathbf{k}_r} = u_{k_1} \wedge \dots \wedge u_{k_r} \in \Lambda^r$, which is called a finite wedge of length r . We also define a finite wedge $u_{\mathbf{k}_r^+} \in \Lambda^r$ for a sufficiently large r by

$$(3.1.2) \quad u_{\mathbf{k}_r^+} = u_{\mathbf{k}_{r_1}^{(1)}}^{(1)} \wedge u_{\mathbf{k}_{r_2}^{(2)}}^{(2)} \wedge \dots \wedge u_{\mathbf{k}_{r_l}^{(l)}}^{(l)}$$

with $u_{\mathbf{k}_{r_i}^{(i)}}^{(i)} = u_{k_1^{(i)}}^{(i)} \wedge \dots \wedge u_{k_{r_i}^{(i)}}^{(i)}$ for $i = 1, \dots, l$, where each r_i is sufficiently large and $r = r_1 + \dots + r_l$. Then in view of 2.3, we see that $u_{\mathbf{k}_r^+}$ is obtained from $u_{\mathbf{k}_r}$ by permuting the sequence \mathbf{k}_r . Moreover, $u_{k_1^{(b)}} \wedge \dots \wedge u_{k_{r_b}^{(b)}}$ is the first r_b -part of the wedge $u_{\mathbf{k}^{(b)}}$ corresponding to $|\lambda^{(b)}, s_b\rangle$ (under the correspondence for the case $l = 1$ in 2.3) for each $b = 1, \dots, l$.

Note that $u_{\mathbf{k}_r^+}$ is not necessarily ordered in general, and it is written as a linear combination of ordered wedges. But the situation becomes drastically simple under the assumption on M -dominance. We have following two lemmas due to Uglov.

Lemma 3.2 ([U, Lemma 5.18]). *Let $b_1, b_2 \in \{1, \dots, l\}$ and $a_1, a_2 \in \{1, \dots, n\}$, and assume that $b_1 < b_2, a_1 \geq a_2$. For any $m \in \mathbb{Z}, t \in \mathbb{Z}_{\geq 0}$, put $X = u_{a_1-nm}^{(b_1)} \wedge u_{a_1-nm-1}^{(b_1)} \wedge \dots \wedge u_{a_1-nm-t}^{(b_1)}$. Then there exists $c \in \mathbb{Z}$ such that the following relation holds.*

$$X \wedge u_{a_2-nm}^{(b_2)} = q^c u_{a_2-nm}^{(b_2)} \wedge X.$$

Lemma 3.3 ([U, Lemma 5.19]). *Take $\lambda \in \Pi^l$ and $\mathbf{s} \in \mathbb{Z}^l(s)$, and assume that $|\lambda, \mathbf{s}\rangle$ is 0-dominant. Then under the notation of 2.3, we have*

$$\begin{aligned} |\lambda, \mathbf{s}\rangle = u_{\mathbf{k}} &= (u_{k_1} \wedge u_{k_2} \wedge \dots \wedge u_{k_r}) \wedge u_{k_{r+1}} \wedge u_{k_{r+2}} \wedge \dots \\ &= q^{-c_r(\mathbf{k})} u_{\mathbf{k}_r^+} \wedge u_{k_{r+1}} \wedge u_{k_{r+2}} \wedge \dots \end{aligned}$$

where $c_r(\mathbf{k}) = \#\{1 \leq i < j \leq r \mid b_i > b_j, a_i = a_j\}$.

3.4. Returning to the setting in 2.11, we describe the map Φ in terms of the wedges. Take $u_{\mathbf{k}} = |\lambda, \mathbf{s}\rangle$, and assume that $|\lambda, \mathbf{s}\rangle$ is 0-dominant. Then by Lemma 3.3, we can write as

$$(3.4.1) \quad u_{\mathbf{k}} = q^{-c_r(\mathbf{k})} (u_{\mathbf{k}_{r_1}}^{(1)} \wedge \cdots \wedge u_{\mathbf{k}_{r_t}}^{(t)}) \wedge (u_{\mathbf{k}_{r_{t+1}}}^{(t+1)} \wedge \cdots \wedge u_{\mathbf{k}_{r_l}}^{(l)}) \wedge u_{r+1} \wedge \cdots,$$

where r_1, \dots, r_l are sufficiently large. Let $u'_{\mathbf{k}'}$ (resp. $u''_{\mathbf{k}''}$) be the ordered wedge in $\Lambda^{s'+\frac{\infty}{2}}$ (resp. in $\Lambda^{s''+\frac{\infty}{2}}$) corresponding to $|\lambda^{[1]}, \mathbf{s}^{[1]}\rangle$ (resp. $|\lambda^{[2]}, \mathbf{s}^{[2]}\rangle$). Since $|\lambda, \mathbf{s}\rangle$ is 0-dominant, $|\lambda^{[i]}, \mathbf{s}^{[i]}\rangle$ is 0-dominant for $i = 1, 2$. Hence again by Lemma 3.3, we have

$$(3.4.2) \quad \begin{aligned} u'_{\mathbf{k}'} &= q^{-c_{r'}(\mathbf{k}')} (u_{\mathbf{k}_{r_1}}'^{(1)} \wedge \cdots \wedge u_{\mathbf{k}_{r_t}}'^{(t)}) \wedge u_{k_{r'+1}}' \wedge \cdots \\ u''_{\mathbf{k}''} &= q^{-c_{r''}(\mathbf{k}'')} (u_{\mathbf{k}_{r_{t+1}}}''^{(t+1)} \wedge \cdots \wedge u_{\mathbf{k}_{r_l}}''^{(l-t)}) \wedge u_{k_{r''+1}}'' \wedge \cdots, \end{aligned}$$

where $r' = r_1 + \cdots + r_t$ and $r'' = r_{t+1} + \cdots + r_l$. Thus, in the case where $|\lambda, \mathbf{s}\rangle$ is 0-dominant, the map $\Phi : u'_{\mathbf{k}'} \otimes u''_{\mathbf{k}''} \mapsto u_{\mathbf{k}}$ is obtained by attaching

$$\begin{aligned} u_{\mathbf{k}_{r_1}}'^{(1)} \wedge \cdots \wedge u_{\mathbf{k}_{r_t}}'^{(t)} &\mapsto u_{\mathbf{k}_{r_1}}^{(1)} \wedge \cdots \wedge u_{\mathbf{k}_{r_t}}^{(t)} \\ u_{\mathbf{k}_{r_{t+1}}}''^{(t+1)} \wedge \cdots \wedge u_{\mathbf{k}_{r_l}}''^{(l-t)} &\mapsto u_{\mathbf{k}_{r_{t+1}}}^{(t+1)} \wedge \cdots \wedge u_{\mathbf{k}_{r_l}}^{(l)} \end{aligned}$$

and by adjusting the power of q .

3.5. We are mainly concerned with the expression as in the right hand side of (3.4.1), instead of treating u_k directly. So we will modify the ordering rule so as to fit the expression by $u_k^{(b)}$. Recall that $u_k^{(b)} = u_{a+n(b-1)-nlm}$ in (3.1.1). We define a total order on the set $\{u_k^{(b)} \mid b \in \{1, \dots, l\}, k \in \mathbb{Z}\}$ by inheriting the total order on the set $\{u_k \mid k \in \mathbb{Z}\} \simeq \mathbb{Z}$. The following property is easily verified.

(3.5.1) Assume that $u_{k_i}^{(b_i)} = u_{a_i+n(b_i-1)-nlm_i}$ for $i = 1, 2$. Then $u_{k_2}^{(b_2)} < u_{k_1}^{(b_1)}$ if and only if one of the following three cases occurs;

- (i) $m_1 < m_2$,
- (ii) $m_1 = m_2$ and $b_1 > b_2$,
- (iii) $m_1 = m_2$, $b_1 = b_2$ and $a_1 > a_2$.

Under this setting, the ordering rule in [U, Prop. 3.16] can be rewritten as follows.

Proposition 3.6. (i) Suppose that $u_{k_2}^{(b_2)} \leq u_{k_1}^{(b_1)}$ for $b_1, b_2 \in \{1, \dots, l\}$, $k_1, k_2 \in \mathbb{Z}$. Let γ be the residue of $k_1 - k_2$ modulo n . Then we have the following formulas.

(R1) the case where $\gamma = 0$ and $b_1 = b_2$,

$$u_{k_2}^{(b_2)} \wedge u_{k_1}^{(b_1)} = -u_{k_1}^{(b_1)} \wedge u_{k_2}^{(b_2)},$$

(R2) the case where $\gamma \neq 0$ and $b_1 = b_2$,

$$\begin{aligned} u_{k_2}^{(b_2)} \wedge u_{k_1}^{(b_1)} &= -q^{-1} u_{k_1}^{(b_1)} \wedge u_{k_2}^{(b_2)} \\ &\quad + (q^{-2} - 1) \sum_{m \geq 0} q^{-2m} u_{k_1 - \gamma - nm}^{(b_1)} \wedge u_{k_2 + \gamma + nm}^{(b_2)} \\ &\quad - (q^{-2} - 1) \sum_{m \geq 1} q^{-2m+1} u_{k_1 - nm}^{(b_1)} \wedge u_{k_2 + nm}^{(b_2)}, \end{aligned}$$

(R3) the case where $\gamma = 0$ and $b_1 \neq b_2$,

$$\begin{aligned} u_{k_2}^{(b_2)} \wedge u_{k_1}^{(b_1)} &= q u_{k_1}^{(b_1)} \wedge u_{k_2}^{(b_2)} \\ &\quad + (q^2 - 1) \sum_{m \geq \varepsilon} q^{2m} u_{k_1 - nm}^{(b_2)} \wedge u_{k_2 + nm}^{(b_1)} \\ &\quad + (q^2 - 1) \sum_{m \geq 1} q^{-2m+1} u_{k_1 - nm}^{(b_1)} \wedge u_{k_2 + nm}^{(b_2)}, \end{aligned}$$

(R4) the case where $\gamma \neq 0$ and $b_1 \neq b_2$,

$$\begin{aligned} u_{k_2}^{(b_2)} \wedge u_{k_1}^{(b_1)} &= u_{k_1}^{(b_1)} \wedge u_{k_2}^{(b_2)} \\ &\quad + (q - q^{-1}) \sum_{m \geq 0} \frac{(q^{2m+1} + q^{-2m-1})}{(q + q^{-1})} u_{k_1 - \gamma - nm}^{(b_1)} \wedge u_{k_2 + \gamma + nm}^{(b_2)} \\ &\quad + (q - q^{-1}) \sum_{m \geq \varepsilon} \frac{(q^{2m+1} + q^{-2m-1})}{(q + q^{-1})} u_{k_1 - nm}^{(b_2)} \wedge u_{k_2 + nm}^{(b_1)} \\ &\quad + (q - q^{-1}) \sum_{m \geq \varepsilon} \frac{(q^{2m} - q^{-2m})}{(q + q^{-1})} u_{k_1 - \gamma - nm}^{(b_2)} \wedge u_{k_2 + \gamma + nm}^{(b_1)} \\ &\quad + (q - q^{-1}) \sum_{m \geq 1} \frac{(q^{2m} - q^{-2m})}{(q + q^{-1})} u_{k_1 - nm}^{(b_1)} \wedge u_{k_2 + nm}^{(b_2)}, \end{aligned}$$

where in the formula (R3) and (R4),

$$\varepsilon = \begin{cases} 1 & \text{if } b_1 < b_2, \\ 0 & \text{if } b_1 > b_2. \end{cases}$$

The sums are taken over all m such that the wedges in the sum remain ordered.

- (ii) For a wedge $u_{k_1}^{(b_1)} \wedge u_{k_2}^{(b_2)} \wedge \cdots \wedge u_{k_r}^{(b_r)}$, above relations hold in every pair of adjacent factors.

Remark 3.7. The ordering rule in the proposition does not depend on the choice of l . It depends only on k_1, k_2 and whether $b_1 = b_2$ or not. This implies the following. Assume that $\mathbf{p} = (t, l - t)$, $s = s' + s''$ and $u_k^{(b)} \in \Lambda^{s' + \frac{\infty}{2}}$, $u_k^{(b)} \in \Lambda^{s'' + \frac{\infty}{2}}$ as before.

Then if $b_1, b_2 \in \{1, 2, \dots, t\}$, the ordering rule for $u_{k_2}^{(b_2)} \leq u_{k_1}^{(b_1)}$ is the same as the rule for $u'_{k_2}^{(b_2)} \leq u'_{k_1}^{(b_1)}$. Similarly, if $b_1, b_2 \in \{t+1, \dots, l\}$ the rule for $u_{k_2}^{(b_2)} \leq u_{k_1}^{(b_1)}$ is the same as the rule for $u''_{k_2}^{(b_2-t)} \leq u''_{k_1}^{(b_1-t)}$.

We show the following three lemmas.

Lemma 3.8. *Let M be an integer such that $M > 2n$. Assume that $u_{\mathbf{k}} = |\lambda, \mathbf{s}\rangle$ is M -dominant. For $b \in \{1, \dots, l\}$ and $i \in \mathbb{Z}$, put $k_i^{(b)} = s_b - i + 1 + \lambda_i^{(b)} = a_i^{(b)} - nm_i^{(b)}$ ($a_i^{(b)} \in \{1, \dots, n\}$, $m_i^{(b)} \in \mathbb{Z}$). Fix $b_1, b_2 \in \{1, \dots, l\}$ such that $b_1 < b_2$. For $k_i^{(b_2)}$, let $\sigma(i)$ be the smallest j such that $u_{k_i^{(b_2)}}^{(b_2)} > u_{k_j^{(b_1)}}^{(b_1)}$. Then we have*

- (i) $\lambda_{\sigma(i)}^{(b_1)} = 0$.
- (ii) $k_{\sigma(i)}^{(b_1)} = n - nm_i^{(b_2)}$.

Proof. Let $\ell(\mu)$ be the number of non-zero parts of a partition μ . We put $p = \ell(\lambda^{(b_1)}) + 1$. In order to show (i), it is enough to see

$$(3.8.1) \quad a_p^{(b_1)} + n(b_1 - 1) - nlm_p^{(b_1)} > a_1^{(b_2)} + n(b_2 - 1) - nlm_1^{(b_2)}$$

In fact, by (3.8.1), we have $u_{k_p^{(b_1)}}^{(b_1)} > u_{k_{\sigma(i)}^{(b_1)}}^{(b_1)}$ since $u_{k_1^{(b_2)}}^{(b_2)} \geq u_{k_i^{(b_2)}}^{(b_2)} > u_{k_{\sigma(i)}^{(b_1)}}^{(b_1)}$. This implies that $p < \sigma(i)$. Since $\lambda_p^{(b_1)} = 0$, we have $\lambda_{\sigma(i)}^{(b_1)} = 0$.

We show (3.8.1). If we put

$$X = (a_p^{(b_1)} + n(b_1 - 1) - nlm_p^{(b_1)}) - (a_1^{(b_2)} + n(b_2 - 1) - nlm_1^{(b_2)}),$$

we have

$$(3.8.2) \quad \begin{aligned} X &= l\{(a_p^{(b_1)} - nm_p^{(b_1)}) - (a_1^{(b_2)} - nm_1^{(b_2)})\} \\ &\quad - (l-1)(a_p^{(b_1)} - a_1^{(b_2)}) - n(b_2 - b_1). \end{aligned}$$

Since

$$\begin{aligned} k_1^{(b_2)} &= s_{b_2} + \lambda_1^{(b_2)} = a_1^{(b_2)} - nm_1^{(b_2)}, \\ k_p^{(b_1)} &= s_{b_1} - \ell(\lambda^{(b_1)}) = a_p^{(b_1)} - nm_p^{(b_1)}, \end{aligned}$$

by replacing $a_1^{(b_2)} - nm_1^{(b_2)}$ by $s_{b_2} + \lambda_1^{(b_2)}$, and similarly for $a_p^{(b_1)} - nm_p^{(b_1)}$ in (3.8.2), we see that

$$\begin{aligned} X &= l\{(s_{b_1} - \ell(\lambda^{(b_1)})) - (s_{b_2} + \lambda_1^{(b_2)})\} - (l-1)(a_p^{(b_1)} - a_1^{(b_2)}) - n(b_2 - b_1) \\ &= l(s_{b_1} - s_{b_2}) - l(\ell(\lambda^{(b_1)}) - \lambda_1^{(b_2)}) - (l-1)(a_p^{(b_1)} - a_1^{(b_2)}) - n(b_2 - b_1) \\ &\geq l(s_{b_1} - s_{b_2}) - l|\lambda| - ln - nl \\ &= l\{(s_{b_1} - s_{b_2}) - (|\lambda| + 2n)\}. \end{aligned}$$

Since $M > 2n$, we have $s_{b_1} - s_{b_2} \geq |\lambda| + M > |\lambda| + 2n$, and so $X > 0$. This proves (3.8.1) and (i) follows.

Next we show (ii). By definition, $\sigma(i)$ is the smallest integer j such that

$$a_i^{(b_2)} + n(b_2 - 1) - nlm_i^{(b_2)} > a_j^{(b_1)} + n(b_1 - 1) - nlm_j^{(b_1)}.$$

If for $a \in \{1, \dots, n\}$ and $m \in \mathbb{Z}$,

$$(3.8.3) \quad a_i^{(b_2)} + n(b_2 - 1) - nlm_i^{(b_2)} > a + n(b_1 - 1) - nlm,$$

then we have $k_p^{(b_1)} > a - nm$ by (3.8.1) and (3.5.1). Note that $k_j^{(b_1)} = a_j^{(b_1)} - nm_j^{(b_1)} = s_{b_1} - j + 1 + \lambda_j^{(b_1)}$ and $\lambda_j^{(b_1)} = 0$ for any $j \geq p$. It follows that $k_{p+j}^{(b_1)} = k_p^{(b_1)} - j$ for any $j \geq 1$. Hence there exists an integer $j_0 \geq 1$ such that $k_{p+j_0}^{(b_1)} = a - nm$. This means that $k_{\sigma(i)}^{(b_1)}$ is the largest integer $a - nm$ for $a \in \{1, \dots, m\}$, $m \in \mathbb{Z}$ satisfying the inequality (3.8.3). Clearly, $a - nm = n - nm_i^{(b_2)}$ is the largest, and we obtain (ii). \square

Lemma 3.9. *For $b_1, b_2 \in \{1, \dots, l\}$ such that $b_1 < b_2$, and $k_1, k_2 \in \mathbb{Z}$ such that $u_{k_2}^{(b_2)} < u_{k_1}^{(b_1)}$, we have*

$$(3.9.1) \quad u_{k_2}^{(b_2)} \wedge u_{k_1}^{(b_1)} = \alpha(k_1, k_2) u_{k_1}^{(b_1)} \wedge u_{k_2}^{(b_2)} + \sum_{\substack{(k'_1, k'_2) \in \mathbb{Z}^2 \\ k_1 > k'_1, k'_2 > k_2}} \alpha(k'_1, k'_2) u_{k'_1}^{(b_1)} \wedge u_{k'_2}^{(b_2)}$$

with $\alpha(k_1, k_2), \alpha(k'_1, k'_2) \in \mathbb{Q}[q, q^{-1}]$.

Proof. Put $k_i = a_i - nm_i$ for $i = 1, 2$, where $a_i \in \{1, \dots, n\}$, $m_i \in \mathbb{Z}$. First assume that $a_1 = a_2$. By the ordering rule (R3) in Proposition 3.6, we have

$$(3.9.2) \quad \begin{aligned} u_{k_2}^{(b_2)} \wedge u_{k_1}^{(b_1)} &= q u_{k_1}^{(b_1)} \wedge u_{k_2}^{(b_2)} \\ &+ (q^2 - 1) \sum_{m \geq 1} q^{2m} u_{k_1 - nm}^{(b_1)} \wedge u_{k_2 + nm}^{(b_2)} \\ &+ (q^2 - 1) \sum_{m \geq 1} q^{-2m+1} u_{k_1 - nm}^{(b_1)} \wedge u_{k_2 + nm}^{(b_2)}, \end{aligned}$$

and the only ordered wedges appear in the sums. Note that $a_1 = a_2$, $b_1 < b_2$ and $m \geq 1$. Then in the second sum, the condition $u_{k_1 - nm}^{(b_1)} > u_{k_2 + nm}^{(b_2)}$ implies that $k_1 > k_1 - nm > k_2 + nm > k_2$ by (3.5.1). It follows that the terms in the second sum are all of the form $u_{k'_1}^{(b_1)} \wedge u_{k'_2}^{(b_2)}$ as in (3.9.1). On the other hand, in the first sum, the condition $u_{k_1 - nm}^{(b_1)} > u_{k_2 + nm}^{(b_2)}$ implies that $k_1 > k_1 - nm \geq k_2 + nm > k_2$ by (3.5.1). Hence if $k_1 - k_2 < 2n$, the terms $u_{k_1 - nm}^{(b_1)} \wedge u_{k_2 + nm}^{(b_2)}$ do not appear in the sum. So assume that $k_1 - k_2 \geq 2n$. We apply the ordering rule (R3) to $u_{k_2 + nm}^{(b_2)} \wedge u_{k_1 - nm}^{(b_1)}$,

and we obtain

$$(3.9.3) \quad u_{k_1-nm}^{(b_2)} \wedge u_{k_2+nm}^{(b_1)} = q^{-1} u_{k_2+nm}^{(b_1)} \wedge u_{k_1-nm}^{(b_2)} + X_1 + X_2,$$

where X_1 (resp. X_2) is a linear combination of the wedges $u_{k_1-nm'}^{(b_2)} \wedge u_{k_2+nm'}^{(b_1)}$ (resp. $u_{k_1-nm'}^{(b_1)} \wedge u_{k_2+nm'}^{(b_2)}$) with $m' > m$. Note that $k_1 > k_2 + nm$ and $k_1 - nm > k_2$, and so $u_{k_2+nm}^{(b_1)} \wedge u_{k_1-nm}^{(b_2)}$ is of the form $u_{k'_1}^{(b_1)} \wedge u_{k'_2}^{(b_2)}$ in (3.9.1). We can apply the same procedure as above for replacing $u_{k_1-nm'}^{(b_2)} \wedge u_{k_2+nm'}^{(b_1)}$ in X_1 by the terms $u_{k'_1}^{(b_1)} \wedge u_{k'_2}^{(b_2)}$ and other terms. Repeating this procedure, finally we obtain the expression as in (3.9.1). Note that since $2n \leq k'_1 - k'_2 < k_1 - k_2$ for $k'_1 = k_1 - nm$ and $k'_2 = k_2 + nm$, this procedure will end up after finitely many steps.

Next consider the case where $a_1 \neq a_2$. In this case we apply the ordering rule (R4). Then one can write as

$$u_{k_2}^{(b_2)} \wedge u_{k_1}^{(b_1)} = u_{k_1}^{(b_1)} \wedge u_{k_2}^{(b_2)} + X_1 + X_2 + X_3 + X_4,$$

where X_1, \dots, X_4 are the corresponding sums in (R4) with $\varepsilon = 1$. For X_1, X_2, X_4 , similar arguments as above can be applied. So we have only to consider the sum X_3 which contains the terms of the form $u_{k_1-\gamma-nm}^{(b_2)} \wedge u_{k_2+\gamma+nm}^{(b_1)}$. In this case, the condition $u_{k_1-\gamma-nm}^{(b_2)} > u_{k_2+\gamma+nm}^{(b_1)}$ implies that $(k_1 - nm) - (k_2 + nm) \geq a_1 - a_2 > -n$, and so $k_1 - k_2 > n(2m - 1)$. Hence, if $k_1 - k_2 \leq n$, the terms $u_{k_1-\gamma-nm}^{(b_2)} \wedge u_{k_2+\gamma+nm}^{(b_1)}$ do not appear in the sum. If $k_1 - k_2 > n$, we can apply a similar argument as before by using the ordering rule (R4). Thus we obtain the expression in (3.9.1) in this case also. The lemma is proved. \square

Lemma 3.10. *Assume that $u_{\mathbf{k}} = |\lambda, \mathbf{s}\rangle$ is M -dominant for $M > 2n$. For $b \in \{1, \dots, l\}$ and $i \in \mathbb{Z}$, put $k_i^{(b)} = s_b - i + 1 + \lambda_i^{(b)}$. Then for $d \in \{t+1, t+2, \dots, l\}$ and $i \in \mathbb{Z}$, we have*

$$\begin{aligned} & u_{k_i^{(d)}}^{(d)} \wedge (u_{k_{r_t}}^{(t)} \wedge u_{k_{r_t-1}}^{(t)} \wedge \dots \wedge u_{k_1}^{(t)} \wedge u_{k_{r_t-1}}^{(t-1)} \wedge \dots \wedge u_{k_2}^{(1)} \wedge u_{k_1}^{(1)}) \\ &= \alpha (u_{k_{r_t}}^{(t)} \wedge u_{k_{r_t-1}}^{(t)} \wedge \dots \wedge u_{k_1}^{(t)} \wedge u_{k_{r_t-1}}^{(t-1)} \wedge \dots \wedge u_{k_2}^{(1)} \wedge u_{k_1}^{(1)}) \wedge u_{k_i^{(d)}}^{(d)} + Y_1, \end{aligned}$$

where $\alpha \in \mathbb{Q}[q, q^{-1}]$ and Y_1 is a $\mathbb{Q}[q, q^{-1}]$ -linear combination of the wedges of the form

$$(u_{\tilde{k}_{r_t}}^{(t)} \wedge u_{\tilde{k}_{r_t-1}}^{(t)} \wedge \dots \wedge u_{\tilde{k}_1}^{(1)}) \wedge u_{\tilde{k}_i}^{(d)} \quad \text{for} \quad (\tilde{k}_{r_t}^{(t)}, \dots, \tilde{k}_1^{(1)}; \tilde{k}_i^{(d)}) \in \mathbb{Z}^{r'} \times \mathbb{Z}$$

under the condition

$$k_{r_t}^{(t)} + \dots + k_1^{(1)} > \tilde{k}_{r_t}^{(t)} + \dots + \tilde{k}_1^{(1)}, \quad k_i^{(d)} < \tilde{k}_i^{(d)}.$$

Proof. Put $k_i^{(d)} = a_i - nm_i$. Let $\sigma(i)$ be the smallest j such that $u_{k_i^{(d)}}^{(d)} > u_{k_j^{(t)}}^{(t)}$. Since $d > t$, by applying Lemma 3.8, we have $k_{\sigma(i)}^{(t)} = n - nm_i$ and $\lambda_{\sigma(i)}^{(t)} = 0$. Thus, we have

$$\begin{aligned} u_{k_i^{(d)}}^{(d)} \wedge (u_{k_{r_t}^{(t)}}^{(t)} \wedge \cdots \wedge u_{k_{\sigma(i)+1}^{(t)}}^{(t)} \wedge u_{k_{\sigma(i)}^{(t)}}^{(t)}) \\ = u_{a_i - nm_i}^{(d)} \wedge (u_{n - nm_i - (r_t - \sigma(i))}^{(t)} \wedge \cdots \wedge u_{n - nm_i - 1}^{(t)} \wedge u_{n - nm_i}^{(t)}). \end{aligned}$$

Using the formula obtained by applying the bar-involution on the formula in Lemma 3.2, we have

$$\begin{aligned} u_{k_i^{(d)}}^{(d)} \wedge (u_{k_{r_t}^{(t)}}^{(t)} \wedge \cdots \wedge u_{k_{\xi(i)+1}^{(t)}}^{(t)} \wedge u_{k_{\sigma(i)}^{(t)}}^{(t)}) \\ = \beta(u_{k_{r_t}^{(t)}}^{(t)} \wedge \cdots \wedge u_{k_{\sigma(i)+1}^{(t)}}^{(t)} \wedge u_{k_{\sigma(i)}^{(t)}}^{(t)}) \wedge u_{k_i^{(d)}}^{(d)} \end{aligned}$$

with $\beta \in \mathbb{Q}[q, q^{-1}]$. Since $u_{k_i^{(d)}}^{(d)} < u_{k_{\sigma(i)-1}^{(t)}}^{(t)} < \cdots < u_{k_1^{(t)}}^{(t)}$, using Lemma 3.9 repeatedly, we have

$$\begin{aligned} u_{k_i^{(d)}}^{(d)} \wedge (u_{k_{r_t}^{(t)}}^{(t)} \wedge \cdots \wedge u_{k_{\sigma(i)+1}^{(t)}}^{(t)} \wedge u_{k_{\sigma(i)}^{(t)}}^{(t)} \wedge u_{k_{\sigma(i)-1}^{(t)}}^{(t)} \wedge \cdots \wedge u_{k_1^{(t)}}^{(t)}) \\ = \beta(u_{k_{r_t}^{(t)}}^{(t)} \wedge \cdots \wedge u_{k_{\sigma(i)+1}^{(t)}}^{(t)} \wedge u_{k_{\sigma(i)}^{(t)}}^{(t)}) \wedge u_{k_i^{(d)}}^{(d)} \wedge (u_{k_{\sigma(i)-1}^{(t)}}^{(t)} \wedge \cdots \wedge u_{k_1^{(t)}}^{(t)}) \\ = \tilde{\beta}(u_{k_{r_t}^{(t)}}^{(t)} \wedge \cdots \wedge u_{k_1^{(t)}}^{(t)}) \wedge u_{k_i^{(d)}}^{(d)} + Y'_1, \end{aligned}$$

where $\tilde{\beta} \in \mathbb{Q}[q, q^{-1}]$ and Y'_1 is a $\mathbb{Q}[q, q^{-1}]$ -linear combination of the wedges of the form

$$(u_{\tilde{k}_{r_t}^{(t)}}^{(t)} \wedge \cdots \wedge u_{\tilde{k}_1^{(t)}}^{(t)}) \wedge u_{\tilde{k}_i^{(d)}}^{(d)} \quad \text{for} \quad (\tilde{k}_{r_t}^{(t)}, \dots, \tilde{k}_1^{(t)}; \tilde{k}_i^{(d)}) \in \mathbb{Z}^{r_t} \times \mathbb{Z},$$

under the condition

$$k_{r_t}^{(t)} + \cdots + k_1^{(t)} > \tilde{k}_{r_t}^{(t)} + \cdots + \tilde{k}_1^{(t)}, \quad k_i^{(d)} < \tilde{k}_i^{(d)}.$$

Thus repeating this procedure for $t-1, \dots, 1$, we obtain the lemma. \square

3.11. We now give a proof of Proposition 2.13. Since $|\lambda, \mathbf{s}\rangle$ is M -dominant, we can write, as in 3.4, that

$$\begin{aligned} |\lambda, \mathbf{s}\rangle &= (u_{k_1} \wedge \cdots \wedge u_{k_r}) \wedge u_{k_r+1} \wedge u_{k_r+2} \wedge \cdots \\ &= \beta(u_{\mathbf{k}_{r_1}^{(1)}}^{(1)} \wedge \cdots \wedge u_{\mathbf{k}_{r_l}^{(l)}}^{(l)}) \wedge u_{k_r+1} \wedge u_{k_r+2} \wedge \cdots, \end{aligned}$$

where $\beta \in \mathbb{Q}[q, q^{-1}]$ and r is sufficient large. By (2.4.1), we have

$$\begin{aligned} \overline{|\lambda, \mathbf{s}\rangle} &= \overline{(u_{k_1} \wedge \cdots \wedge u_{k_r})} \wedge u_{k_r+1} \wedge u_{k_r+2} \wedge \cdots \\ &= \overline{\beta \left(u_{\mathbf{k}_{r_1}^{(1)}}^{(1)} \wedge \cdots \wedge u_{\mathbf{k}_{r_l}^{(l)}}^{(l)} \right)} \wedge u_{k_r+1} \wedge u_{k_r+2} \wedge \cdots. \end{aligned}$$

By (2.4.2), we have

$$\begin{aligned} \overline{u_{\mathbf{k}_{r_1}^{(1)}}^{(1)} \wedge \cdots \wedge u_{\mathbf{k}_{r_l}^{(l)}}^{(l)}} \\ = \beta' \left(u_{k_{r_l}^{(l)}}^{(l)} \wedge \cdots \wedge u_{k_1^{(t+1)}}^{(t+1)} \right) \wedge \left(u_{k_{r_t}^{(t)}}^{(t)} \wedge \cdots \wedge u_{k_2^{(1)}}^{(1)} \wedge u_{k_1^{(1)}}^{(1)} \right) \end{aligned}$$

with $\beta' \in \mathbb{Q}[q, q^{-1}]$. By using Lemma 3.10 repeatedly, we have

$$\begin{aligned} & \left(u_{k_{r_l}^{(l)}}^{(l)} \wedge \cdots \wedge u_{k_2^{(t+1)}}^{(t+1)} \wedge u_{k_1^{(t+1)}}^{(t+1)} \right) \wedge \left(u_{k_{r_t}^{(t)}}^{(t)} \wedge \cdots \wedge u_{k_2^{(1)}}^{(1)} \wedge u_{k_1^{(1)}}^{(1)} \right) \\ &= \beta'' \left(u_{k_{r_g}^{(t)}}^{(t)} \wedge \cdots \wedge u_{k_2^{(1)}}^{(1)} \wedge u_{k_1^{(1)}}^{(1)} \right) \wedge \left(u_{k_{r_l}^{(l)}}^{(l)} \wedge \cdots \wedge u_{k_2^{(t+1)}}^{(t+1)} \wedge u_{k_1^{(t+1)}}^{(t+1)} \right) + Y, \end{aligned}$$

where $\beta'' \in \mathbb{Q}[q, q^{-1}]$, and Y is a $\mathbb{Q}[q, q^{-1}]$ -linear combination of the wedges of the form

$$\left(u_{\tilde{k}_{r_t}^{(t)}}^{(t)} \wedge \cdots \wedge u_{\tilde{k}_1^{(1)}}^{(1)} \right) \wedge \left(u_{\tilde{k}_{r_l}^{(l)}}^{(l)} \wedge \cdots \wedge u_{\tilde{k}_1^{(t+1)}}^{(t+1)} \right) \quad \text{for} \quad \begin{cases} (\tilde{k}_{r_t}^{(t)}, \dots, \tilde{k}_1^{(1)}) \in \mathbb{Z}^{r'}, \\ (\tilde{k}_{r_l}^{(l)}, \dots, \tilde{k}_1^{(t+1)}) \in \mathbb{Z}^{r''} \end{cases}$$

under the condition

$$(3.11.1) \quad \begin{aligned} k_1^{(1)} + \cdots + k_{r_t}^{(t)} &> \tilde{k}_1^{(1)} + \cdots + \tilde{k}_{r_t}^{(t)}, \\ k_1^{(t+1)} + \cdots + k_{r_l}^{(l)} &< \tilde{k}_1^{(t+1)} + \cdots + \tilde{k}_{r_l}^{(l)}. \end{aligned}$$

We claim that

(3.11.2) The wedges appearing in Y is written as a linear combination of the wedges $u_{\mathbf{h}} = |\mu, \mathbf{s}\rangle$ such that $\mathbf{a}(\lambda) > \mathbf{a}(\mu)$ for $\mu \in \Pi^l$.

We show (3.11.2). By using the ordering rule, $u_{\tilde{k}_{r_t}^{(t)}}^{(t)} \wedge \cdots \wedge u_{\tilde{k}_1^{(1)}}^{(1)}$ (resp. $u_{\tilde{k}_{r_l}^{(l)}}^{(l)} \wedge \cdots \wedge u_{\tilde{k}_1^{(t+1)}}^{(t+1)}$) can be written as a linear combination of the wedges $u_{\mathbf{h}_{r_1}^{(1)}}^{(1)} \wedge \cdots \wedge u_{\mathbf{h}_{r_t}^{(t)}}^{(t)}$ (resp. $u_{\mathbf{h}_{r_{t+1}}^{(1)}}^{(1)} \wedge \cdots \wedge u_{\mathbf{h}_{r_l}^{(l)}}^{(l)}$) with $u_{\mathbf{h}_{r_i}^{(i)}}^{(i)} = u_{h_1^{(i)}}^{(i)} \wedge \cdots \wedge u_{h_{r_i}^{(i)}}^{(i)}$. Proposition 3.6 says that the sum of the indices is stable in applying the ordering rule. Hence we have

$$(3.11.3) \quad \begin{aligned} \tilde{k}_1^{(1)} + \cdots + \tilde{k}_{r_t}^{(t)} &= h_1^{(1)} + \cdots + h_{r_t}^{(t)}, \\ \tilde{k}_1^{(t+1)} + \cdots + \tilde{k}_{r_l}^{(l)} &= h_1^{(t+1)} + \cdots + h_{r_l}^{(l)}. \end{aligned}$$

Recall that $k_i^{(b)} = s_b - i + 1 + \lambda_i^{(b)}$. We define $\mu \in \Pi^l$ by setting $h_i^{(b)} = s_b - i + 1 + \mu_i^{(b)}$ for any i, b , and write it as $\mu = (\mu^{[1]}, \mu^{[2]})$. Then (3.11.1) and (3.11.3) imply that $|\lambda^{[1]}| > |\mu^{[1]}|$. Also we note that $|\lambda| = |\mu|$ since

$$k_1^{(1)} + \cdots + k_{r_l}^{(l)} = \tilde{k}_1^{(1)} + \cdots + \tilde{k}_{r_l}^{(l)} = h_1^{(1)} + \cdots + h_{r_l}^{(l)}.$$

Hence we have $\mathbf{a}(\lambda) > \mathbf{a}(\mu)$. Moreover, by using (3.4.1), we see that $(u_{\mathbf{h}_{r_1}^{(1)}}^{(1)} \wedge \cdots \wedge u_{\mathbf{h}_{r_l}^{(l)}}^{(l)}) \wedge u_{k_{r+1}} \wedge \cdots$ coincides with $u_{\mathbf{h}} = |\mu, \mathbf{s}\rangle$ up to scalar. Thus (3.11.2) is proved.

Now as noted in Remark 3.7, the ordering rule for $u_{k_{r_t}}^{(t)} \wedge \cdots \wedge u_{k_1^{(1)}}^{(1)}$, regarded as an element in $\Lambda^{s+\frac{\infty}{2}}$ or as an element in $\Lambda^{s'+\frac{\infty}{2}}$, is the same. Hence under the map Φ , $u_{k_{r_t}}^{(t)} \wedge \cdots \wedge u_{k_1^{(1)}}^{(1)}$ (resp. $u_{k_{r_t}}^{(t)} \wedge \cdots \wedge u_{k_1^{(1)}}^{(t+1)}$) corresponds to $u_{k_{r_t}}^{(t)} \wedge \cdots \wedge u_{k_1^{(1)}}^{(1)}$ (resp. $u_{k_{r_t}}^{(t)} \wedge \cdots \wedge u_{k_1^{(1)}}^{(t+1)}$). It follows that

$$\begin{aligned} & (u_{k_{r_t}}^{(t)} \wedge \cdots \wedge u_{k_1^{(1)}}^{(1)}) \wedge (u_{k_{r_l}}^{(l)} \wedge \cdots \wedge u_{k_1^{(t+1)}}^{(t+1)}) \wedge u_{r+1} \wedge u_{r+2} \wedge \cdots \\ &= \alpha \overline{|\lambda^{[1]}, \mathbf{s}^{[1]}\rangle} \otimes \overline{|\lambda^{[2]}, \mathbf{s}^{[2]}\rangle} \end{aligned}$$

with some $\alpha \in \mathbb{Q}[q, q^{-1}]$. Summing up the above arguments, we have

$$(3.11.4) \quad \overline{|\lambda, \mathbf{s}\rangle} = \alpha \overline{|\lambda^{[1]}, \mathbf{s}^{[1]}\rangle} \otimes \overline{|\lambda^{[2]}, \mathbf{s}^{[2]}\rangle} + \sum_{\substack{\mu \in \Pi^l \\ \mathbf{a}(\lambda) > \mathbf{a}(\mu)}} \alpha_{\lambda, \mu} |\mu, \mathbf{s}\rangle$$

with $\alpha, \alpha_{\lambda, \mu} \in \mathbb{Q}[q, q^{-1}]$. Since the coefficient of $|\lambda, \mathbf{s}\rangle$ in the expansion of $\overline{|\lambda, \mathbf{s}\rangle}$ in terms of the ordered wedges is equal to 1, and similarly for $|\lambda^{[1]}, \mathbf{s}^{[1]}\rangle, |\lambda^{[2]}, \mathbf{s}^{[2]}\rangle$, we see that $\alpha = 1$ by comparing the coefficient of $|\lambda, \mathbf{s}\rangle = |\lambda^{[1]}, \mathbf{s}^{[1]}\rangle \otimes |\lambda^{[2]}, \mathbf{s}^{[2]}\rangle$ in the both sides of (3.11.4). This proves the proposition.

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